An exact solution for flux pileup merging in a curved current sheet

Yuri Litvinenko

Department of Mathematics, University of Waikato, New Zealand
Motivation

- Exact MHD solutions for magnetic reconnection illustrate its key features—a small thickness of the current sheet, flux pileup at the entrance to the sheet, Alfvénic outflows.

- Analytical solutions have been discovered for flux pileup merging and reconnection in 2D and 3D (Clarke 1964; Parker 1973; Sonnerup & Priest 1975; Craig & Henton 1995; Craig & Fabling 1996 ... Craig & Litvinenko 2012).

- All analytical models describe current sheets sustained by a stagnation-point flow. Can analytical solutions based on other types of flows be constructed?

- Yes.
Governing MHD equations

Momentum equation:
\[
\partial_t v + (v \cdot \nabla)v = -\nabla p + (\nabla \times B) \times B + \nu \nabla^2 v
\]

Induction equation:
\[
\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B
\]

Incompressibility equation:
\[
\nabla \cdot v = 0
\]

No-name equation:
\[
\nabla \cdot B = 0
\]

Units: \( v_A = B_0 / \sqrt{4\pi \rho_0} = 1 \), \( L = 1 \), \( \nu = 1 / \text{Re} \), \( \eta = 1 / \text{Lu} \).
The form of the solution

Axisymmetric stagnation flow on a cylinder (Wang 1974):

\[ \mathbf{v} = -\alpha \left( r - \frac{a^2}{r} \right) \hat{r} + 2\alpha z \hat{z} \]

(1) A source if \( \alpha > 0 \), a sink if \( \alpha < 0 \);
    a stagnation-point flow if \( a = 0 \).

(2) Work is underway on incorporating vorticity.

Steady magnetic field:

\[ \mathbf{B} = B(r) \hat{z} \]
The pressure profile

\[ \nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 0, \quad \nabla^2 \mathbf{v} = 0, \]

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = -B'(r) \hat{\theta}, \quad (\nabla \times \mathbf{B}) \times \mathbf{B} = -BB' \hat{\mathbf{r}}. \]

Momentum equation \[ \rightarrow \]

\[ p(r, z) = \text{const} - \frac{1}{2} v^2 - \frac{1}{2} B^2 \]
The merging magnetic field

Induction equation \[ \eta(rB' - aJ) + \alpha(r^2 - a^2)B = 0 \]

\[ E = \eta J \frac{a}{r} \hat{\theta}, \quad J = B'(a) \]

The limit \( a = 0 \): magnetic tube

\[ B(r) = B(0) \exp\left(-\alpha r^2 / 2\eta\right) \]

(Moffatt 1978)

MHD Burgers vortex with vorticity \( \hat{\omega} = \omega \hat{z} \)

(Kambe 1985)
The magnetic field profile

\[ B(r) = r^{\alpha a^2/\eta} \exp \left( -\frac{\alpha r^2}{2\eta} \right) f(r) \]

\[ f(r) = a J \int r^{-1-\alpha a^2/\eta} \exp \left( \frac{\alpha r^2}{2\eta} \right) dr \]

\[ B(r) = \left[ C r^{\alpha a^2/\eta} - \frac{a J}{2} \left( -\frac{\alpha r^2}{2\eta} \right)^{\alpha a^2/2\eta} \Gamma \left( -\frac{\alpha a^2}{2\eta}, -\frac{\alpha r^2}{2\eta} \right) \right] \times \exp \left( -\frac{\alpha r^2}{2\eta} \right) \]

\[ B(0) = -\frac{\eta J}{\alpha a}, \quad B(a) = 0 \]
Merging magnetic field profile

Parameters: $\alpha = 0.5$, $\eta = 0.5$, $a = 3$, $B(a) = 0$, $J = 1$
The effect of curvature is small if $r \approx a$, so $B(r)$ near the sheet has the same functional form as in the planar geometry solution (Sonnerup & Priest 1975; Craig & Henton 1995).
Current sheet scalings

Inner solution for $B(r)$:

$$B_i(r) \approx J(r - a)$$

Outer solution for $B(r)$:

$$B_o(r) \approx \frac{a \eta J}{\alpha (r^2 - a^2)}$$

$B_i(l) \approx B_o(l) \implies \text{sheet thickness}$

$$l \approx \left(\frac{\eta}{2\alpha}\right)^{1/2} \sim \eta^{1/2}$$

Magnetic field at the entrance to the sheet: $B_s \approx Jl$
Fast merging?

Strong flux pileup \((B_s > 1)\) is required if the merging rate \(\eta J\) is to exceed the Sweet–Parker rate \(E \approx \eta^{1/2}\).

\[
E(a) = \eta J \approx 1 \implies B_s \sim \eta^{-1/2}.
\]

Stagnation on a cylinder \(\implies\) larger energy release rate

Volume in which the magnetic energy release takes place:

\[
a(\eta/\alpha)^{1/2} \gg \eta/\alpha
\]

(in comparison with Moffatt’s tube or Craig’s spine merging).
Anisotropic (parallel) viscosity

Viscous force = \frac{\partial S_{ij}}{\partial x_j}

Parallel (Braginskii) viscosity:

\[ S_{ij} = \nu \left( \delta_{ij} - 3 \frac{B_i B_j}{B^2} \right) \left( \frac{B_k B_l}{B^2} \frac{\partial v_k}{\partial x_l} - \frac{1}{3} \nabla \cdot \mathbf{v} \right) \]

\[ S_{xx} = S_{yy} = \nu \frac{\partial v_z}{\partial z} = 2\alpha \nu, \quad S_{zz} = -2S_{xx}. \]

\frac{\partial S_{ij}}{\partial x_j} = 0 \implies \text{the viscous force vanishes whether the plasma viscosity is isotropic or anisotropic}

(but viscous heating, ultimately caused by unbalanced viscous stresses at the boundaries, is generally present, e.g., Hollweg 1986).
Photospheric canceling features (BBSO)

(Litvinenko & Martin 1999)
Photospheric magnetic reconnection

(Sturrock 1999)
Geometry of photospheric cancellation (2D)

(Litvinenko & Martin 1999)
Geometry of photospheric cancellation (3D)

(Litvinenko & Martin 1999)
Photospheric reconnecting current sheet

(Litvinenko, Chae, & Park 2007)
Photospheric canceling features (SOHO MDI)

(Chae, Moon, & Park 2003)
Application to photospheric flux cancellation

A curved photospheric current sheet
Scalings for a canceling magnetic feature

Model parameter $\alpha$:

$$\alpha = \frac{v_i \cos(L/2a)}{a \sin^2(L/2a)}.$$  

Thickness of the current sheet:

$$\frac{l}{a} \approx \left[ \frac{\eta \sin^2(L/2a)}{2av_i \cos(L/2a)} \right]^{1/2}.$$
Summary

An exact magnetohydrodynamic solution is presented for steady magnetic annihilation (merging) in an incompressible resistive viscous plasma.

The merging, driven by an axisymmetric stagnation flow on a cylinder, takes place in a curved current sheet that is perpendicular to the plane in which the plasma flow stagnates. The new solution extends earlier models of flux pileup merging in a flat current sheet, driven by stagnation-point flows.

The new solution remains valid in the presence of both the isotropic and anisotropic (parallel) plasma viscosity. The geometry of the solution may make it useful in modeling the photospheric flux cancellation on the Sun.