WEAKLY INHOMOGENEOUS MHD TURBULENCE AND TRANSPORT OF SOLAR WIND FLUCTUATIONS

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Abstract

Certain observed properties of solar wind fluctuations require descriptions outside the traditional WKB or homogeneous turbulence frameworks. This article presents a brief review of recent theories of transport of small scale MHD turbulence in an inhomogeneous background which show promise for providing more complete explanations of the evolution of solar wind turbulence.

1 Introduction

A useful way of looking at solar wind MHD fluctuations is to first ask how the fluctuations behave locally in space and time, say, on the scale of a few correlation lengths and times, and second, to ask how this characterization changes as the fluctuations evolve under the influence of the slowly varying background. For example, because the mean wind is both supersonic and super-Alfvénic, the radial evolution of the fluctuations corresponds to a temporal evolution of each outward moving blob of magnetofluid. The first of these questions has often been addressed by treating the fluctuations as propagating MHD waves, mostly Alfvén waves, a view supported both by minimum variance arguments and by high correlations of plasma velocity and magnetic field fluctuations in the inner heliosphere /1/. Alternatively, a viewpoint that the fluctuations are a form of actively evolving MHD turbulence affords an appealing mechanism for plasma heating in the solar wind /2/, while also providing a framework for a reasonably consistent interpretation of a variety of detailed statistical properties of the observations /3/. The question of the large scale transport and evolution of the fluctuations has traditionally been addressed in the context of WKB theory /4,5,6,7/. This approach works fairly well to explain the radial evolution of the wave energy density, but fails in a number of important ways. In this paper we review briefly the current status of recently developed theories of the spatial and wavenumber transport of MHD *turbulence* in the weakly inhomogeneous solar wind that may provide explanations of a number of observed phenomena that cannot be treated using wave-based WKB transport.

2 Solar Wind Spectral Evolution in WKB and related theories

The traditional way to write transport equations for the evolution of the fluctuation power spectra in the solar wind begins with the WKB formalism for MHD waves in a weakly inhomogeneous background /4,5,6,7/. In the lowest order WKB approximation one finds that

$$\partial_t P^{\pm}(k) + L^{\pm}_{WKB} P^{\pm}(k) = 0, \qquad (1)$$

where $P^{\pm}$ are the reduced (or one-dimensional) power spectra of inward ($-$) and outward ($+$) propagating waves, with argument wavenumber k. We can denote the WKB spatial transport operator as $L^{\pm}_{WKB} = L^{\pm}_{x} + 2L^{\pm}$, with $L^{\pm}_{x} \equiv (\mathbf{U} \mp \mathbf{V}_A) \cdot \nabla$ and $L^{\pm} \equiv \nabla \cdot (\mathbf{U}/4 \pm \mathbf{V}_A/2)$ The conditions for validity of the WKB expansion (discussed further below) include the enforcement of both the wave dispersion relation, and the scale separation condition $kR \gg 1$.

This WKB formalism has been employed in many solar wind and other space physics applications, and it is widely believed to account reasonably well for certain observed phenomena, such as the radial evolution of the fluctuation level δB^2 in the outer (> 1AU) heliosphere. However, there are also known shortcomings. For example, it is a consequence of the wave description that there exists no "mixing" of Alfvénic fluctuations at the leading order, i.e., the equations for the development of the + and − fluctuations are not coupled. Although mixing does nevertheless occur at higher orders /8/, this mixing is insufficient /9/ to account for the evolution of cross-helicity with radial distance in the solar wind /10/. A similar statement can be made regarding the behavior of the "Alfvén ratio", i.e. the ratio of kinetic to magnetic energy per unit mass in inertial range fluctuations. Observations /3,10/ indicate that the Alfvén ratio is often somewhat less than unity in the solar wind, but it remains exactly one in leading order WKB theory.

Further shortcomings of WKB theory involve its inability to (a) account for the rapid evolution of the spectral shape of solar wind fluctuations in the inner heliosphere /10,11/ and (b) provide a rapid wave damping mechanism that would address the source of heating of solar wind plasma /12/ between the lower corona and 1 AU. Coleman /2/ had suggested that turbulent heating, at a rate such as that predicted for homogeneous turbulence in Kolmogoroff theory /13/, might provide the required mechanism.

In a pair of important and seminal papers, Tu and coworkers /14,15/, combined Coleman's heating suggestion with WKB transport, opening the way for more complete treatments of the turbulence. The essence of Tu's models is to combine leading order WKB spatial transport with a simple phenomenology for spectral transfer in wavenumber, giving rise to a transport equation

$$\partial_t P^+(k) + L^+_{WKB} P^+(k) = NL^+, \tag{2}$$

for the spectrum of outward propagating fluctuations. The term NL^+ in (2) represents triple correlations arising from nonlinear terms in the MHD equations. The first model /14/ adopted what is essentially the Kraichnan /16/ phenomenology of spectral transfer in the inertial range for modeling NL^+, while the second /15/ utilized the Kolmogoroff /13/ inertial range phenomenology. In each case the nonlinearities are written as $NL = \partial G/\partial k$ where the energy flux in wavenumber space, G, is approximated by dimensional analysis and the usual statistical assumptions applied to the turbulent inertial range. A number of similar approaches can be adopted in modeling nonlinearities responsible for local turbulence /17/. The Tu theories for evolution of the spectrum have enjoyed some success in accounting for modifications to the energy spectrum in the inner heliosphere. However, because the inward and outward waves are still uncoupled, they cannot describe nontrivial dynamics of either the cross helicity or the Alfvén ratio.

An equally important advance in solar wind transport, was made /18,19/ in connection with dynamical transport equations for the total wave energy. Phenomenological theories of *total* turbulent energy decay in homogeneous turbulence /13/ make use of the global eddy turnover time as an estimate of the decay time of the energy-containing eddies. Hollweg /18,19/ adopted this "Kolmogoroff"perspective, along with a WKB approximation for the spatial transport of the total wave energy, in a model for the acceleration and heating of the solar wind in the inner heliosphere. While such a procedure directly addresses the ideas set forth by Coleman /2/, the predictions of this theory /20/ have not yet been able to account simultaneously for both the wave energies and the temperatures at 1 AU, when reasonable parameter values at the coronal critical points are used.

3 Structure of two-scale transport theories

The models of Tu and Hollweg provide ample motivation to develop more comprehensive theories for transport of solar wind turbulence. One would like to describe in such a theory not only the radial evolution of turbulent energy in both the energy-containing and inertial ranges, but also the possibility of variations in the cross helicity content, the interaction of "inward" and "outward" type fluctuations, and unequal values of kinetic and magnetic fluctuation energies. There exists also the possibility of studying transport of other spectral quantities, such as magnetic helicity and induced electric field. Moreover, there is a basic conceptual question that arises in both the Tu and Hollweg models: If the turbulence is "strong" enough to be treated by Kolmogoroff theory, how can the dispersion relation underlying WKB theory also be enforced? In the last several years, these questions have been investigated by appeal to more general formalisms for transport of turbulence /21,22,23,24/ based mainly upon the assumptions of scale separation and local incompressibility. The simplest forms of the transport theory assume the large scale plasma velocity and magnetic fields are specified.

The theory can be cast into two forms, one appropriate to spectral evolution in the inertial range /21,22,23/ and the other to the evolution of the energy-containing eddies near the correlation scale /25/. In each case, dynamical equations based on a two length scale expansion are derived, from which the evolution of various quantities may be computed, including magnetic and kinetic energies, cross helicity, induced electric field, and the corresponding helicities.

By the assumption of scale separation, we mean that the turbulent fluctuations, consisting of fluctuations at scales up to a correlation length λ, admit locally well-defined statistical properties that vary on the scale R that characterizes changes in the inhomogeneous background velocity, magnetic and density fields. Taking R to be the local heliocentric distance, this amounts to the assertion that $\epsilon = \lambda/R$ is a small parameter. Fast and slow-varying space coordinates, are introduced to separate local effects (e.g., turbulent spectral transfer) and effects associated with inhomogeneities. Similarly, fast and slow time scales can also be introduced. Thus, for position $\mathbf{r}$ and time t, we let $\mathbf{r} = \mathbf{x}$, $t = \tau$ (slow scale) and $\mathbf{x}' = \mathbf{x}/\epsilon$ and $t' = t/\epsilon$ (fast scale). To facilitate calculations, we introduce an averaging operator $<\ldots>$ that annihilates fast scale variations. For example, the solar wind (proton) fluid velocity $\mathbf{V}$ may be decomposed into $\mathbf{V} = \mathbf{U} + \mathbf{u}$ where $\mathbf{U} = <\mathbf{V}>$ is the mean large scale flow and $\mathbf{u}$ is the fluctuating velocity. Similarly, the magnetic field can be separated according to $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$, with $\mathbf{B_0} = <\mathbf{B}>$ and turbulent fluctuation $\mathbf{b}$. In the simplest cases, the density ρ is taken to be locally incompressible, $\rho = <\rho>$ and the mean fields $\mathbf{U}$, $\mathbf{B_0}$ and ρ are assume to be specified, time independent functions varying only with heliocentric distance R. It is also convenient to work in Elsässer variables for the fluctuations, $\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{b}/\sqrt{4\pi\rho}$.

Transport equations are computed by forming the correlation functions $R_{ij}^{++} = < z_i^+ z_j^{+\prime} >$, $R_{ij}^{--} = < z_i^- z_j^{-\prime} >$ and $R_{ij}^{+-} = < z_i^+ z_j^{-\prime} >$, and making use of the compressible MHD equations and the assumption of statistical homogeneity on the fast scales. In the above the primed ($'$) and unprimed variables are evaluated at distinct spatial positions and the associated separation vector (generally in the radial direction for solar wind spacecraft observations) is the sole dependency of the correlation matrices. The associated spectral tensors, i.e., the Fourier transform of the correlation matrices, are denoted as S_{ij}^{++}, S_{ij}^{--} and S_{ij}^{+-} having wavevector $\mathbf{k}$ as argument. The procedure leads to equations involving the operators introduced earlier, $L_x^{\pm}$ and $L^{\pm}$, which appeared in WKB theory. Also appearing is a new matrix operator $M_{ij}^{\pm}$ that involves only derivatives of the large scale fields, and which gives rise to "mixing" type couplings between inward and outward type fluctuations /21,22,26/. The explicit form of the spectral equations so obtained is quite complex /22,23/ and will not be repeated here. However, the general form of the equations is $(\mathbf{1}\partial_t + \mathbf{L}_S) \cdot \mathbf{s} = \mathbf{nl}$ where $\mathbf{L}_S$ is a linear spatial transport operator The elements s_i of the solution vector $\mathbf{s}$ are the wavenumber dependent scalar spectra, and the right hand side is a column vector $\mathbf{nl}$ whose elements are the modeled nonlinear terms associated with the corresponding spectra.

In the most general case of homogeneous turbulence, it can be shown that the maximum number of independent elements s_i is 16. Four of these are identified with the antisymmetric parts of the spectral tensors and can be represented by the magnetic helicity, the helicity of the velocity field, the "helicity of the cross helicity" and the single spectral scalar that generates the induced electric field spectrum /22/. Of the 12 spectra contained in the symmetric parts of the three spectral tensors, six arise from the three independent elements of each of the symmetric energy spectral tensors $S_{ij}^{++}+S_{ji}^{++}$ and $S_{ij}^{--}+S_{ji}^{--}$. The remaining six are accounted for by the three independent elements of the energy difference tensor (kinetic minus magnetic) and the three elements connected with the "helicity of the electric field"/22/.

For a particular symmetry of the turbulence, a natural choice of independent spectral scalars is usually evident. A selection of slab, two-dimensional (2D) or isotropic turbulence forces equality between certain of the scalars, and causes others to vanish. Moreover, it is sometimes convenient to write the spectral equations in terms of scalar spectra that depend upon wave vector **k**, retaining the full three dimensional domain of the vector argument. In other cases it may prove better to write everything in terms of one dimensional, "reduced" spectra that are easily connected with single spacecraft solar wind observations /3/. In the latter case especially, the symmetry of the turbulence enters in a crucial fashion, since various symmetry-dependent relationships between modal, omnidirectional and reduced spectra /13/ need to be used to simplify the final equations. For typical cases, including slab, 2D and isotropic turbulence, it is convenient to write the spectral transport equations in a form including energy spectra of the $\pm$ fields $P^{\pm}(k)=S_{ii}^{\pm\pm}/4$, the difference of kinetic and magnetic energy spectra $F(k)=S_{ii}^{+-}/4$, and so on.

The spatial transport operator $\mathbf{L}_S$ includes familiar effects such as convection and expansion associated with the solar wind velocity, and wave propagation in the direction of (or opposite to) the large scale Alfvén velocity. For the symmetries mentioned above, the transport equation for the $\pm$ energies has the form

$$\partial_t P^{\pm}(k)+L_{WKB}^{\pm}P^{\pm}(k)+M^{\pm}F=NL^{\pm}, \tag{3}$$

where further equations for F, etc. are required to close the model. The new term $M^{\pm}$ represents leading order mixing, and has a form specific to the choice of symmetry /22,26/. Space prohibits an exhaustive treatment of either the structure or solutions of the full transport theory in this paper. The model remains a subject of intense investigation and comparison with solar wind observations. Thus, we restrict our comments here to several pertinent general points.

- The spectral evolution theory consists of up to 16 coupled equations, which can be simplified by approximations in special cases. For isotropic turbulence, the number of coupled equations may be as many as seven, or, with suitable approximations regarding the behavior of F, as few as two. A three equation isotropic model, involving $P^{\pm}$ and F, appears to be physically reasonable /26/.

- One interesting limit /27/ is that of WKB theory, and convergence to that case is demonstrated for strong Alfvén speed and weak nonlinearities, i.e, enforcement of the wave dispersion relation /26/. Strong "mixing" requires nontrivial couplings of $P^{\pm}$ with F. The distinction between strong mixing and recovery of WKB results is explored further below.

- Several simple analytic solutions /21,22/ predict, with increasing heliocentric distance, a decrease of the preponderance of outward-traveling type Alfvénic fluctuations and a lowering of the small-scale kinetic to magnetic energy ratio. Both of these are roughly consistent with Helios and Voyager observations /10/.

- The theory requires, for closure, modeling of the nonlinear terms. For spectral quantities conserved in the inertial range at high Reynolds numbers, such as $P^{\pm}(k)$, correct models will adopt nonlinearities of the form $NL^{\pm}=\partial G^{\pm}(k)/\partial k$, where $G^{\pm}$ is an appropriately defined wavenumber flux of the associated conserved energy. In strong turbulence such terms vanish. For nonconserved spectral quantities such as $F(k)$, this type of model is inappropriate, and in strong turbulence the associated transport effects may not vanish. Turbulence theory has yet to provide us with well-accepted forms for model nonlinearities. Howver, simple approaches, including diffusion in k-space, one point closures and simple relaxation time models may be useful /17,24,28/ for empirical and observational comparisons.

- The spectral transport model can also be developed further into a model for the evolution of the energy containing fluctuations, in analogy with classical quasi-equilibrium range hydrodynamic turbulence /13/ and making use of phenomenologies of homogeneous MHD turbulence. A closed five equation model for evolution of the turbulence has been proposed /25/, including transport and nonlinear evolution of bulk magnetic energy, kinetic energy, cross helicity and two correlation scales. The energy-containing model differs from the spectral transport model above mainly in the way in which the nonlinearities are handled, and is expected to be useful in providing the required low wavenumber boundary data for the spectral transport model that operates in the inertial range. In fact, it is expected /13/ that inertial range energy transfer rates, and thus the overall heating rate, will be regulated mainly by the decay of the energy containing eddies.

4 Relation to WKB theory: Multiple scales analysis

Some of the consequences of the multi-equation, scale separated transport models /21,22,23/ may be disconcerting, especially in that couplings between inward and outward-type fluctuations, and therefore the"mixing" effect, appear in the leading order theory. How is it that such effects are absent in WKB theory but present here? Is an error implied in the classical derivations of WKB transport? Nonlinearities do not easily explain these discrepancies, since the mixing effect occurs in the *linear* transport terms. These concerns /27/ have been discussed in connection with the spectral transport theory, neglecting nonlinearities, the suggestion emerging that within *linear* theory, the mixing effect should be treated in leading order for certain cases. Specifically, when $\mathbf{k}\cdot\mathbf{V}_A \to 0$ or when $\mathbf{V}_A/U \to 0$, the strong mixing effect is present in leading order, owing to the degeneracy of the two solutions to the wave dispersion relation in those limits. It is also possible to reconcile WKB and a non-WKB "mixing theory" entirely in the context of the primitive linear equations for the fluctuating fields. We outline this procedure here.

Let the fluctuating Elsässer fields be expanded as $\mathbf{z}^{\pm}(\mathbf{x},\mathbf{x}',\tau,\tau') = \mathbf{z}^{\pm 0} + \epsilon\mathbf{z}^{\pm 1} + \epsilon^2\mathbf{z}^{\pm 2} + \cdots$. The leading order ϵ^{-1} expansion yields

$$\left(\partial_{\tau'} + L^{\pm}_{\mathbf{x}'}\right) z_i^{\pm 0} = 0, \tag{4}$$

where the primed operators involve derivatives with respect to the fast scale. The solution to (4) is written as $\mathbf{z}^{\pm 0} = \tilde{\mathbf{z}}^{\pm 0} e^{iS_\pm}$, in which $\tilde{\mathbf{z}}^{\pm 0}$ is slowly varying envelope function and $S_\pm$ (the eikonal or phase function) depends on both fast and slow coordinates. By proper choice of the (arbitrary) slowly varying dependence of $S_\pm$, one can simplify subsequent manipulations. This is accomplished by demanding that the phase functions themselves obey the ϵ^{-1} equation with the fast derivatives replaced by slow derivatives.

Proceeding to the $O(1)$ expansion and making use of the leading order solution yields

$$\left(\partial_{\tau'} + L^{\pm}_{\mathbf{x}'}\right) z_i^{\pm 1} = -e^{iS_\pm}\left[\left(\partial_\tau + L^{\pm}_{\mathbf{x}} + L^{\pm}\right)\tilde{z}_i^{\pm 0}\right] - e^{iS_\mp} M^{\pm}_{ik}\tilde{z}_k^{\mp 0}. \tag{5}$$

This equation is an inhomogeneous wave equation of the same type as the leading order $O(\epsilon^{-1})$ wave equation, therefore the solution consists of a particular plus homogeneous solution $z^{\pm 1} = z_p^{\pm 1} + z_h^{\pm 1}$. Clearly, $z_h^{\pm 1}$ has the same form (eikonal) as the zeroth order term, hence $z_h^{\pm 1} = \tilde{z}_h^{\pm 1} e^{iS_\pm}$. In determining the particular solution, care must be taken to avoid secularities. This is equivalent to avoiding resonances in the inhomogeneous wave equation (5). Such considerations lead to two distinct choices of solvability condition, one of which leads back to well-known classical WKB theory, while the other corresponds to the non-WKB multiple scales approach with strong mixing.

To arrive at the WKB solvability condition, assume that the $\mathbf{z}^+$ inhomogeneity is non-resonant with $\mathbf{z}^-$. At this order, the assumption is equivalent to $S_+ \neq S_-$. Hence the only restriction on the inhomogeneous wave equation (5) is that

$$\left(\partial_\tau + L^{\pm}_{\mathbf{x}} + L^{\pm}\right)\tilde{z}_0^{\pm} = 0.$$

This is the leading order WKB equation for the primitive fields $\mathbf{z}^{\pm}$, and implies WKB transport for the energy spectra as given in (1).

The non-WKB solvability criterion is obtained through the assumption that $\mathbf{z}^{+}$ and $\mathbf{z}^{-}$ are nearly resonant in the sense that $S_{+} \approx S_{-}$. Hence, the solvability condition for (5) is that the full RHS vanish. We now have no inhomogeneities and the leading order slowly varying amplitudes obey the equation

$$(\partial_{\tau} + L^{\pm}_{\mathbf{x}} + L^{\pm})\, \bar{z}^{\pm 0} + \epsilon_0 M^{\pm} \bar{z}^{\pm}_{0} = 0,$$

where $\epsilon_0 = e^{i(S_{-}-S_{+})}$. This represents a non-WKB form of the transport equation with mixing possible at the leading order, and is equivalent to (3) without the nonlinear terms. In general, the size of the mixing term depends on the magnitude of ϵ_0 which in the present case is $O(1)$. It is apparent that it is necessary to choose the non-WKB conditions when $S_{+} \approx S_{-}$, which is equivalent to either of the conditions $\mathbf{k} \cdot \mathbf{V}_A \approx 0$ or $V_A/U \rightarrow 0$. These are two of the conditions identified previously /27/ for leading order "mixing" in linear spectral transport equations.

5 Conclusions

We have briefly reviewed the background and basic principles leading to the recent development of transport equations for MHD turbulence in the weakly inhomogeneous background solar wind. Various approximate theories based on this approach show promise in answering important questions in heliospheric physics. Future applications in lower coronal physics, in shock theory and in astrophysics are also feasible. In a new development given in Sec. 4, we have outlined the mathematical connection between WKB and non-WKB theories, further implications of which will be presented at a future opportunity.

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6 References

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