Is the Alfvén-wave propagation effect important for energy decay in homogeneous MHD turbulence?

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Abstract. We investigate the role of three-point decorrelation due to Alfvén wave propagation in three-dimensional incompressible homogeneous MHD turbulence. By comparing numerical simulations with theoretical expectations, we have studied how this effect influences the decay of turbulent energy caused by both an external mean magnetic field and the fluctuating turbulent field. Decay is initially suppressed by a mean magnetic field, as expected, but the effect soon saturates. The decay rate does not scale with mean magnetic field for higher values. The disagreement with theoretical predictions can be accounted for by anisotropic spectral transfer. Thus, phenomenological models for energy decay that include decorrelation due to Alfvénic propagation are not substantiated. This work complements our detailed study of various models of energy decay in homogeneous MHD [*Hossain et al.*, 1995].

Introduction

MHD turbulence is recognized as being of central importance to numerous subjects of ongoing research, e.g., the sun, the heliosphere, space plasmas, and cosmic rays. While these applications may eventually require precise accounts of the dynamical evolution of turbulence, an immediate need is to include reasonable quantitative estimates of MHD turbulence in dynamical models appropriate to the specific application. The simplest such models involve phenomenological treatments of the energy decay, which can be used to estimate heating rates in solar wind transport models.

We start with the incompressible MHD equations involving the solenoidal fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$. The magnetic field is written as the sum of a locally uniform mean value $\mathbf{B}_0$ and a fluctuating part $\mathbf{b}$. Because mass density is constant, the equations are cast in Alfvén speed units [*Fyfe and Montgomery*, 1976], where magnetic field variables are equivalent to their associated Alfvén velocity, i.e., $(4\pi\rho)^{-1/2}\mathbf{B}_0 \rightarrow \mathbf{B}_0 = \mathbf{V}_A$. The dynamical equations are conveniently written using *Elsässer* [1950] variables $\mathbf{z}_\pm = \mathbf{u} \pm \mathbf{b}$ as

$$\frac{\partial \mathbf{z}_\pm}{\partial t} = -\mathbf{z}_\mp \cdot \nabla \mathbf{z}_\pm \pm \mathbf{V}_A \cdot \nabla \mathbf{z}_\pm - \mathbf{S}_\pm. \qquad (1)$$

The pressure term acts on very short timescales to ensure solenoidal vector fields and will be ignored. Dissipation due to viscosity and resistivity, important mainly at small scales, is represented by $\mathbf{S}_\pm$.

Decay Rates and Propagation Effects

For homogeneous MHD turbulence we are particularly interested in the dynamical behavior of the fluctuation energy per unit mass, $E = \langle |\mathbf{u}|^2 + |\mathbf{b}|^2 \rangle/2 = \langle |\mathbf{z}_+|^2 + |\mathbf{z}_-|^2 \rangle/4$, and the fluctuation cross helicity density, $H_c = \langle \mathbf{u} \cdot \mathbf{b} \rangle$. Angle brackets represent volume averages. These quantities are rugged invariants of the ideal MHD equations and particularly relevant to the theory of turbulent spectral transfer [*Kraichnan*, 1973;*Frisch et al.*, 1975; *Stribling and Matthaeus*, 1991]. The amount of E or H_c present is not modified by the nonlinear terms that mediate spectral transfer among eddies of different wavenumbers. In the high Reynolds number limit, these quantities can change only when excitations reach the dissipation region at very small scales. The so called Elsässer energies $Z_+^2 = \langle |\mathbf{z}_+|^2 \rangle$ and $Z_-^2 = \langle |\mathbf{z}_-|^2 \rangle$ are equivalent invariants.

Kraichnan [1965] and *Dobrowolny et al.* [1980b] applied the approach of *Karman* [1938] and *Kolmogorov* [1941a,b] for hydrodynamic models to the turbulent decay of the Elsässer energies $Z_\pm^2$ in MHD. For the present purposes, the first step is to write the decay of $Z_\pm^2$ in terms of spectral transfer times $\tau_s^\pm$,

$$\frac{dZ_\pm^2}{dt} = -\alpha_\pm \frac{Z_\pm^2}{\tau_s^\pm}. \qquad (2)$$

The parameters $\alpha_\pm$ are phenomenological constants of order unity; the degree to which they do remain constant in the simulations indicates the validity of a conjectured $\tau_s^\pm$. The spectral transfer times are estimated from several characteristic time scales arising from the the dynamical equations (1). The first term on the right yields $\tau_{nl}^\pm$, the characteristic nonlinear time generalized for MHD in Elsässer representation [*Dobrowolny et al.*, 1980],

$$\tau_{nl}^\pm = \frac{\lambda_\pm}{Z_\mp}. \qquad (3)$$

This time scale represents the approximate rate at which an eddy at the energy-containing length scale $\lambda_\pm$ decays due to correlations with turbulent structures of the other sense. Consequently, the typical lifetime of these correlations $\tau_3^\pm$ (called triple correlations, after the Fourier representation of the term) is also important. The spectral, nonlinear, and triple-decorrelation timescales are related by [*Matthaeus and Zhou*, 1989]:

$$\tau_s^\pm = \frac{(\tau_{nl}^\pm)^2}{\tau_3^\pm}. \qquad (4)$$

In addition, energy decay has been argued to depend upon wave-propagation effects arising from the second term and represented by a characteristic Alfvénic time scale $\tau_A^{\pm}$. The superscript $\pm$ on each of these variables admits the possibility that they are generally different for the two Elsässer fields.

Each of these quantities has been discussed previously [*Kraichnan*, 1965; *Grappin et al.*, 1982, 1983; *Grappin et al.*, 1983; *Matthaeus and Zhou*, 1989], but mainly in the context of inertial-range phenomenology, and in essentially an approximation of isotropic turbulence. It is not immediately clear how these inertial-range arguments should be applied in the energy-containing range for phenomenological models of MHD energy decay.

For steady inertial-range energy transfer, *Kraichnan* [1965] argued that the triple correlation lifetime τ_3 should be dominated by Alfvénic decorrelation when propagation is sufficiently strong that $\tau_A \ll \tau_{nl}$. He suggested that τ_A should be the period of Alfvén waves of the appropriate scale in the mean magnetic field. *Pouquet et al.* [1976] suggested that inertial-range triple correlations also decay due to propagation in the fluctuating magnetic field. Under the isotropy assumption, the wave period at wavenumber k is of order $(kV_A)^{-1}$. These ideas can be directly carried over to the energy-containing range by using $\lambda_{\pm}$ in place of k^{-1}. This leads to a characteristic Alfvén time

$$\tau_A^{\pm} = \frac{\lambda_{\pm}}{\sqrt{V_A^2 + b^2}}, \tag{5}$$

where b is the rms magnitude of $\mathbf{b}$.

In general, triple correlations also decay due to the nonlinear process characterized by τ_{nl}, but the *Kraichnan* [1965] phenomenology neglects this effect because the mean magnetic field strength is assumed to be large. For cases where both effects are important, the total rate of triple decorrelation is plausibly given by the sum of the two rates [*Matthaeus and Zhou*, 1989]:

$$\frac{1}{\tau_3^{\pm}} = \frac{1}{\tau_{nl}^{\pm}} + \frac{1}{\tau_A^{\pm}}. \tag{6}$$

Mean Field Effect

For strong mean magnetic field with $Z_{\pm} \ll B_0$, the extension of *Kraichnan*'s [1965] model into the energy-containing range predicts an energy decay rate of

$$\frac{dZ_{\pm}^2}{dt} = -\alpha_{\pm}\frac{Z_+^2 Z_-^2}{\lambda_{\pm} B_0}. \tag{7}$$

This suggests that decay should be suppressed by increasing the mean magnetic field. To evaluate this prediction, we plot the time histories of energy for three comparable simulations with varied mean fields in Figure 1. The two runs with $B_0 = 3$ and 8 clearly decay slower than the $B_0 = 0$ run, but the suppression seems to have saturated with $B_0 = 3$. If (7) were true, then the decay rate in the $B_0 = 8$ case would be almost three times smaller than in the $B_0 = 3$ case. Obviously, this does not happen in the simulations (at least for the relatively low Reynolds number cases reported here), so the Alfvén wave decorrelation hypothesis is incorrect in the energy-containing range.

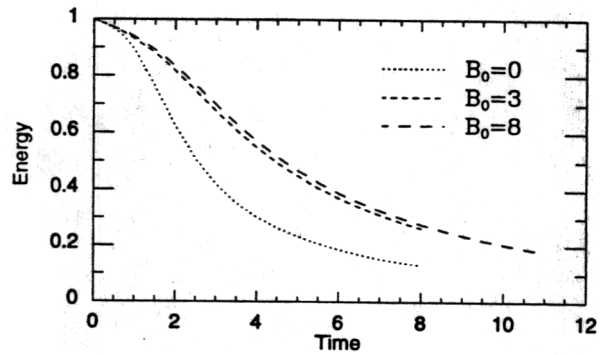

Figure 1. Evolution of total energy for three runs with varying mean magnetic field B_0. Note the suppression of decay as B_0 increases from 0 to 3 and the comparatively negligible change when B_0 is increased to 8.

Fluctuating Field Effect

To examine possible effects of wave propagation due to the fluctuating magnetic field, we treat examples with no mean magnetic field. In that case $\tau_A^{\pm} = \lambda_{\pm}/b$ and

$$\frac{dZ_{\pm}^2}{dt} = -\frac{\alpha_{\pm}}{\lambda_{\pm}}\frac{Z_{\pm}^2 Z_{\mp}^2}{Z_{\mp} + b} \tag{8}$$

We will evaluate this equation by testing the constancy of $\alpha_{\pm}$. First we consider the case when the cross helicity and energy difference D are zero (i.e., equipartition in kinetic and magnetic energies), so that $Z_+ = Z_- \equiv Z$ and $b^2 = Z_+^2/2 = Z_-^2/2 = Z^2/2$. We also now have only one length scale λ, so that $\tau_A^{\pm} = \sqrt{2}\lambda/Z$. Therefore, the decay model with wave propagation effect due to the fluctuating magnetic field is:

$$\frac{dZ^2}{dt} = -\frac{1}{(1+\sqrt{2})}\alpha\frac{Z^3}{\lambda} \tag{9}$$

This suggests that the effect of wave propagation would be to scale the decay rate by a simple numerical constant of order unity. Given the inherent uncertainty in the correct value of $\alpha_{\pm}$, there does not appear to be a practical way to distinguish the two theories in this situation.

To demonstrate the level at which the models are successful, we begin by computing the right-hand side of the decay equation (8) with the propagation term b first included then omitted. These values are compared to the decay rate on the left-hand side determined by taking finite differences in the simulations. The ratio of the model value for dZ_+^2/dt to that determined from the simulation can be thought of as a time-dependent value of α_+, which is supposed constant in the models. Figure 2 shows two values of computed α_+ (with and without the propagation term b) for a run started from $D = 0$ and $Z_+ = Z_-$. After an initial transient period,

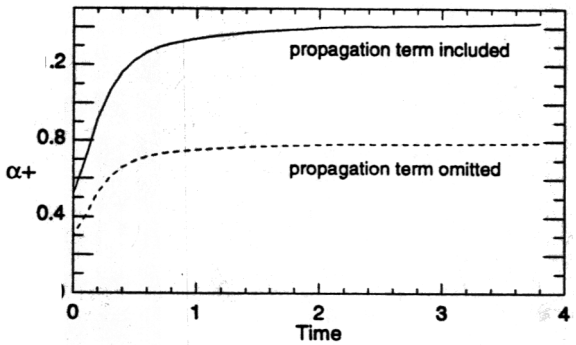

Figure 2. Computed α_+ from simulation data for zero initial cross helicity and energy difference.

both curves approach level asymptotes, verifying that either model reasonably represents the dynamic evolution of the turbulence.

To distinguish between the two models, we turn to a case with large cross helicity. Consider a situation with $D = 0$ again, but let cross helicity be so large that $Z_+ \ll Z_-$ and $b \lesssim Z_+/2$. In that case $\tau_A^{\pm} \approx 2\lambda_{\pm}/Z_+$, so the decay equation becomes:

$$\frac{dZ_{\pm}^2}{dt} \approx -\alpha_{\pm} \frac{Z_{\pm}^2 Z_{\mp}^2}{\lambda_{\pm}(Z_{\mp} + Z_+/2)}. \tag{10}$$

By putting almost all the fluctuation energy into the Z_+ field, the symmetry between the $\pm$ equations has been broken. For the Z_+ field,

$$\frac{dZ_+^2}{dt} = -\alpha_+ \frac{Z_+^2 Z_-}{\lambda_+} \left[\frac{2Z_-}{Z_+}\right], \tag{11}$$

where the term in brackets appears only when the propagation term is included. This extreme example suggests that simulations with variable cross helicity should yield profiles of α_+ that plateau better for one model than for the other. No pronounced difference is expected for the minority fluctuations, apart from a constant of order unity between the α_- values as in the previous test.

We now evaluate the models against a simulation with nonzero initial cross helicity. The ratio Z_+/Z_- is initially 1.5 and rises to 4.5 by $t = 8$. Figure 3 shows the values of $\alpha_{\pm}$ computed from equation (8) with and without the propagation term, as before. Between $t = 1$ and 8, the case with propagation rises by a factor of 4, while the other remains within a factor of about 2 of unity. While neither asymptotes as definitively as the zero helicity case, it is clear that the model without propagation is significantly better. There remains room for improvement, but including propagation does not appear to be a step in the right direction. The α_- profiles generally support either model to the same degree of confidence and confirm that the two behave similarly to each other, as suggested by our simple analysis.

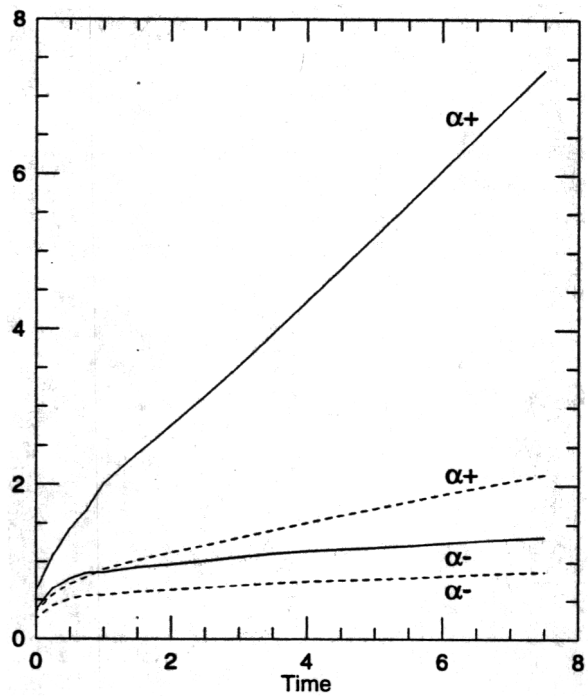

Figure 3. Similar to Figure 2, only a significant and variable cross helicity is present in these simulations to distinguish between the model predictions. The α_+ curves have qualitatively different behaviors that demonstrate the advantage of the model without propagation (dashed curve) compared to that with it (solid).

Summary and Discussion

These simulations suggest that the Alfvén decorrelation effect associated with the propagation of small-scale structures along the large-scale fluctuating magnetic field does not contribute to the overall decay of the energy. This implies that the mechanism proposed by *Pouquet et al.* [1976] for the inertial range does not apply for the energy-containing range. Dropping this effect returns to a fluid-like model for the decay of the energy-containing eddies in MHD turbulence. A class of such models and their properties have been examined in a recent paper [*Hossain et al.*, 1995].

Regarding the Alfvén wave propagation effect due to an external mean magnetic field, we showed that the decay rate does not scale with the strength of the mean field as predicted by adapting *Kraichnan*'s [1965] model to the energy-containing range. However, for finite mean field the decay is indeed diminished from that when it is zero, an effect that has been attributed to anisotropic spectral transfer [*Hossain et al.*, 1995]. Numerical simulations of both freely decaying [*Shebalin et al.*, 1983; *Carbone and Veltri*, 1990; *Oughton et al.*, 1994] and driven [*Hossain et al.*, 1985] MHD turbulence have shown that wavenumber spectra become distinctly anisotropic in the presence of a uniform mean magnetic

field of sufficient strength. This anisotropy is consistent with relatively rapid spectral transfer of energy to wave vectors that are nearly perpendicular to the mean field, with accordingly slower transfer to those aligned with the mean field. This phenomenon has been described in terms of a resonance argument [*Shebalin et al.*, 1983].

Models that try to include the effect of Alfvénic propagation assume that all wave-vectors experience the full decorrelation effect of the mean field $\mathbf{B}_0$, as though each is aligned with the mean field. The increased anisotropy due to enhanced perpendicular transfer evidently tends to suppress Alfvénic decorrelation effects. In the inertial range this effect modifies the effective Alfvén time from $\tau_A(\mathbf{k}) = (kB_0)^{-1}$, the value appropriate for slab turbulence, to $|\mathbf{k}\cdot\mathbf{B}_0|^{-1}$, so that the orientation of $\mathbf{k}$ enters explicitly. An anisotropic version of τ_s is also appropriate in the energy-containing range. One simple approach would be to modify the spectral transfer rate (cf. (4)) for anisotropic turbulence to read

$$\frac{1}{\tau_s^{\pm}} = \frac{Z_{\mp}^2}{\lambda_{\pm}}\frac{1}{(Z_{\mp} + B_0\cos\theta_{\pm})}, \qquad (12)$$

where $\cos\theta_{\pm}$ measures the degree of anisotropy [*Shebalin et al.*, 1983] of the $z_{\pm}$ spectrum in the energy-containing range. Accounting for anisotropy by using the spectral transfer rate (12) to modify the decay equations (2) has been successful [*Hossain et al.*, 1995].

The straightforward extension of *Kraichnan*'s [1965] inertial-range phenomenology to the energy-containing range does not work as well as a simple energy decay model of the Karman-Kolmogoroff type. That is, there appears to be no explicit dependence of the modeled spectral transfer or the decay rates upon either the large-scale Alfvén speed (for large enough B_0) or that associated with the rms value of $\mathbf{b}$. The mean magnetic field does influence the simulation results, however, and a better phenomenological model can be obtained by taking spectral anisotropy into account [*Hossain et al.*, 1995]. This model has the energy decay equations

$$\frac{dZ_{\pm}^2}{dt} = -\frac{1}{A}\frac{Z_{\pm}^2 Z_{\mp}}{\lambda_{\pm}}, \qquad (13)$$

where $A \approx 1$ when $B_0 \approx 0$, and $A \approx 2$ when $B_0 \geq 1$. Under suitable conditions this parameter emulates the effects of spectral anisotropy.

In closing, we note that one might expect Alfvén wave propagation effects to remain valid for inertial-range dynamics. But if spectral transfer through the inertial range into the dissipation range is reduced by whatever mechanism, then that effect should slow the decay of the total turbulent energy and be reflected in the evolution of the energy-containing eddies. Because no such effect is evident, the conclusions of this study for the energy-containing eddies suggest that the wave-propagation effect may not be operative in the inertial range. Although propagation effects may strongly influence the angular distribution of energy flux in the inertial range, our results imply they do so without changing the total energy flux.

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