Impact of Hall effect on energy decay in magnetohydrodynamic turbulence

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[1] We examine numerically the influence of Hall effect corrections to Ohm's law upon the decay of homogeneous compressible magnetohydrodynamic turbulence and conclude that there are *no significant differences in global decay rate associated with the Hall effect*. This affirms expectations that energy decay is controlled by the large-scale eddies. *INDEX TERMS:* 2149 Interplanetary Physics: MHD waves and turbulence; 2752 Magnetospheric Physics: MHD waves and instabilities; 7827 Space Plasma Physics: Kinetic and MHD theory; 7863 Space Plasma Physics: Turbulence; 7867 Space Plasma Physics: Wave/particle interactions. **Citation:** Matthaeus, W. H., P. Dmitruk, D. Smith, S. Ghosh, and S. Oughton, Impact of Hall effect on energy decay in magnetohydrodynamic turbulence, *Geophys. Res. Lett.*, *30*(21), 2104, doi:10.1029/2003GL017949, 2003.

[2] It is of considerable importance in the study of astrophysical and space-plasma turbulence to understand the regulation of the rate of decay of turbulent energy, and the associated rate of heat deposition. Simulations indicate [*Hossain et al.*, 1995; *Mac Low*, 1999] that in collisional homogeneous magnetohydrodynamic (MHD) turbulence, decay of energy is self-similar and hydrodynamic-like in that the dissipation rate [*von Kármán and Howarth*, 1938] is controlled by the large-scale eddies. While the underpinnings of this viewpoint are based upon collisional models, the same approach accounts well for radial evolution [*Zank et al.*, 1996] and associated heating [*Matthaeus et al.*, 1999] of collisionless solar wind turbulence. Of various kinetic corrections to MHD, the Hall effect [*Krall and Trivelpiece*, 1973; *Turner*, 1983] has emerged as being of particular importance in recent studies of collisionless magnetic reconnection. Some evidence [*Birn et al.*, 2001] suggests that the Hall effect is required to correctly simulate reconnection rates [*Shay et al.*, 1998]. On this basis there appear to be two possibilities regarding the impact of Hall effect on energy decay in MHD turbulence.

[3] First, one can argue [*von Kármán and Howarth*, 1938; *Hossain et al.*, 1995; *Mac Low*, 1999] that the large eddies control the dissipation rate and the specific mechanism for small-scale dissipation is not of central importance. Large-scale fluctuations control the cascade, and fast, small-scale dissipation mechanisms, whether collisional or not, act as passive absorbers of cascaded energy. Consequently, the decay rate of energy would be unaffected by inclusion of Hall current effects.

[4] A second line of reasoning rests on the observation that MHD turbulence proceeds through successive, scale-invariant reconnection events [*Matthaeus and Lamkin*, 1986]. Therefore, if Hall-MHD gives reconnection rates that are distinctly different from ordinary viscous-resistive MHD, the cascade rate might be affected, and hence the global decay rate of energy would be modified in the collisionless Hall regime. A clear dichotomy thus is presented, and in this Letter we report results of numerical simulations designed to directly examine this issue.

[5] Decay of fluctuation energy is greatly enhanced in turbulence compared with laminar flow due to the involvement of a wide range of spatial scales—in turbulence dissipation is typicaly enhanced by a factor $(k_d\lambda)^2 \gg 1$ where λ is the energy-containing scale and k_d is the dissipation wavenumber. Self-similarity of the correlation functions [*von Kármán and Howarth*, 1938; *Mac Low*, 1999] implies that $k_d \sim \nu^{-1/2}$. Thus, small-scale gradients are amplified until the dissipation rate becomes insensitive to the value of the viscosity ν. As a consequence, for undriven turbulence the overall decay of turbulent energy is regulated by the large-scale energy-containing eddies.

[6] The Hall effect enters as a correction to Ohm's law for low density plasmas. In laboratory units, in terms of resistivity η, electric current density $\mathbf{j}$, plasma velocity $\mathbf{v}$, magnetic field $\mathbf{B}$, electric field $\mathbf{E}$, electric charge e, speed of light c, and number density n, Ohm's law becomes

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \eta \mathbf{j} = \frac{\mathbf{j} \times \mathbf{B}}{nec}, \tag{1}$$

where electron pressure and electron inertia are neglected [*Krall and Trivelpiece*, 1973]. The Hall effect term [r.h.s. of equation (1)] becomes important at lengths comparable to the ion inertial scale $\rho_{ii} = c/\omega_{pi}$ (plasma frequency ω_{pi}), which in low-collisionality astrophysical plasmas is typically $\ll\lambda$, but $\rho_{ii} > \lambda_{diss} \sim 1/k_d$, the latter being the scale at which dissipation is effective. Therefore the Hall effect may modify the nonlinear cascade. Recent ("GEM") studies of collisionless reconnection [*Birn et al.*, 2001] emphasize that the Hall term breaks the ideal MHD property that the

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magnetic field is frozen into the plasma. Consequently, the Hall effect might modify the cascade by facilitating small-scale reconnection events, and this may change the energy decay rate even though Hall effect does not directly cause dissipation.

[7] The physics of interest is contained in a one-fluid compressible 3D MHD model, involving the fluctuating fluid velocity $\mathbf{v}(x, y, z, t)$, density $\rho(x, y, z, t)$ and a magnetic field $\mathbf{B}(x, y, z, t) = \mathbf{b}(x, y, z, t) + \mathbf{B}_0$, that may include a uniform constant $\mathbf{B}_0$. The standard turbulence units are based on the mean initial fluctuation speed U_0, the characteristic length scale $L \sim \lambda$ on the order of the simulation box size (energy-containing scale), and the mean density ρ_0. The normalized Hall-MHD equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho \gamma M_s^2} \nabla P + \frac{\mathbf{j} \times \mathbf{B}}{\rho} + \frac{1}{R} D(\mathbf{v}), \tag{3}$$

$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{v} \times \mathbf{B} - \epsilon_H \frac{\mathbf{j} \times \mathbf{B}}{\rho} + \frac{1}{R_m} \nabla^2 \mathbf{a}, \tag{4}$$

where $\mathbf{j} = \nabla \times \mathbf{B}$, Alfvén speed units are employed ($\mathbf{B} \rightarrow \mathbf{B}/\sqrt{4\pi\rho_0}$), $\nabla \times \mathbf{a} = \mathbf{b}$, and $\nabla \cdot \mathbf{a} = 0$. The large-scale Reynolds number $R = 1/\nu$ and magnetic Reynolds number $R_m = 1/\mu$ are reciprocals of normalized kinematic viscosity ν and magnetic diffusivity μ. In equation (3), $D(\mathbf{v}) = (\nabla^2 \mathbf{v} + \frac{1}{3}\nabla\nabla \cdot \mathbf{v})/\rho$. The Mach number is $M_s = U_0/c_s$, with $c_s^2 = \partial P/\partial \rho|_{\rho_0}$. The Hall parameter is $\epsilon_H = \rho_{ii}/L$.

[8] For the present purposes it suffices to have polytropic pressure, $P \sim \rho^\gamma$ ($\gamma = 5/3$) and to restrict attention to the small M_s nearly incompressible regime [*Zank and Matthaeus*, 1993]. Neglect of electron pressure in Ohm's law is justified if gradients of the electron pressure are not too large. The present results are intended for such cases, and this approximation is favored by near-incompressibility. Fluctuation amplitudes (exclusive of small compressive contributions) are conveniently measured by the kinetic energy per unit mass $\langle |\mathbf{v}|^2 \rangle/2$ and the magnetic energy per unit mass $\langle |\mathbf{b}|^2 \rangle/2$, the brackets denoting a volume average. For convenience, we define $E^{tot} = \langle \mathbf{v}^2 + \mathbf{b}^2 \rangle/2$.

[9] The nonlinear MHD cascade generates progressively smaller scale fluctuations until scales at which dissipation is effective are reached. We ask whether the decay of fluctuations differs when the Hall term is present ($\epsilon_H \neq 0$) versus when it is absent, varying ϵ_H from unity, down through small values to zero. When $\epsilon_H = \rho_{ii}/L \ll 1$, as is typical for astrophysical and space plasmas, the Hall term is negligible at large scales and contributes significantly at small scales $\leq \rho_{ii}$. In the solar wind $\epsilon_H \approx 10^{-4}$ while in the lower solar corona $\epsilon_H \approx 10^{-5}$. (See, e.g., parameters in *Axford and McKenzie* [1997] and discussion of astrophysical applications in *Mininni et al.* [2003].)

[10] Our present studies focus on constant density initial states with $E^{tot} = 1$ and low cross helicity $H_c = \langle \mathbf{v} \cdot \mathbf{b} \rangle$ and magnetic helicity $H_m = \langle \mathbf{a} \cdot \mathbf{b} \rangle$. Note that with ϵ_H and $\mathbf{B}_0$ nonzero, H_m and H_c are no longer conserved in the ideal equations, but are replaced by a generalized magnetic helicity [*Matthaeus and Goldstein*, 1982a] and generalized hybrid helicity [*Turner*, 1983; *Mahajan and Yoshida*, 2000]; for simplicity we retain a description in terms of the former quantities. For each such initial state we perform various runs, altering ϵ_H and/or B_0. The Mach number is fixed at a low value, $M_s = 0.25$. Nonzero fluctuation amplitudes are initially equipartitioned and random-phased in the $\mathbf{k}$-space (wavevector) shell $1 \leq |\mathbf{k}| \leq 4$. The duration of the runs is 10–20 nominal nonlinear time units, by which time the energy has decayed substantially. To solve equations (2)–(4) we employ a triply periodic (side $2\pi L$) Fourier pseudo-spectral code [*Ghosh et al.*, 1993, 1996], adapted for use on a Beowulf cluster [*Dmitruk et al.*, 2001]. We report here on results from runs with resolutions of 64^3, 128^3, and 256^3 Fourier modes, with $R = R_m = 1000$ for 256^3 and 128^3 runs, and $R = R_m = 400$ for 64^3 runs. This scheme is non-diffusive (no numerical dissipation of energy), second-order in time, and ensures exact energy conservation for the continuous time spatially discrete equations. In principle, stabilized aliasing and infinite order convergence of this method (see *Ghosh et al.* [1993]) render it equivalent to a much large number of finite difference points. No artificial viscosity or stabilizing filters are employed.

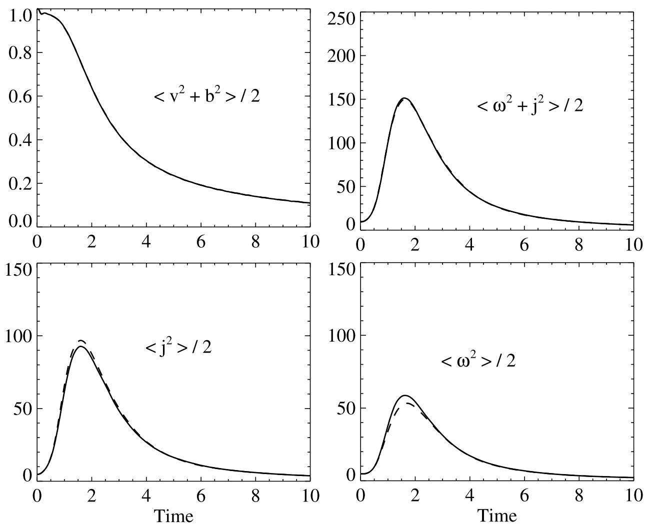

Figure 1. 256^3 MHD simulations. Top left: fluctuation energy with Hall effect, $\epsilon_H = 1/32$ (dashed line), and without Hall effect, $\epsilon_H = 0$ (solid line); Right: mean-square current plus mean-square vorticity. Bottom left: mean-square electric current density; Right: mean-square vorticity.

[11] Figure 1 compares representative $B_0 = 0$ simulations, having $\epsilon_H = 1/32$ and 0, showing the time history of the fluctuation energy E^{tot}. There is very little difference between runs with and without Hall effect. Also shown is the sum of mean-square current $\langle \mathbf{j}^2 \rangle$ and mean-square vorticity $\langle \omega^2 \rangle$ vs. time, where $\omega = \nabla \times \mathbf{v}$. Only relatively small differences are seen, mainly near the time of maximum small-scale activity, at around 1–2 nonlinear times. It is subsequent to this maximum of dissipation rate that one expects the onset of self-similar decay.

[12] The difference in total decay rate is small, but closer examination reveals subtle effects of the Hall term at high k. In Figure 1, we separate the decay rate for this run into its contributions from mean-square current and mean-square

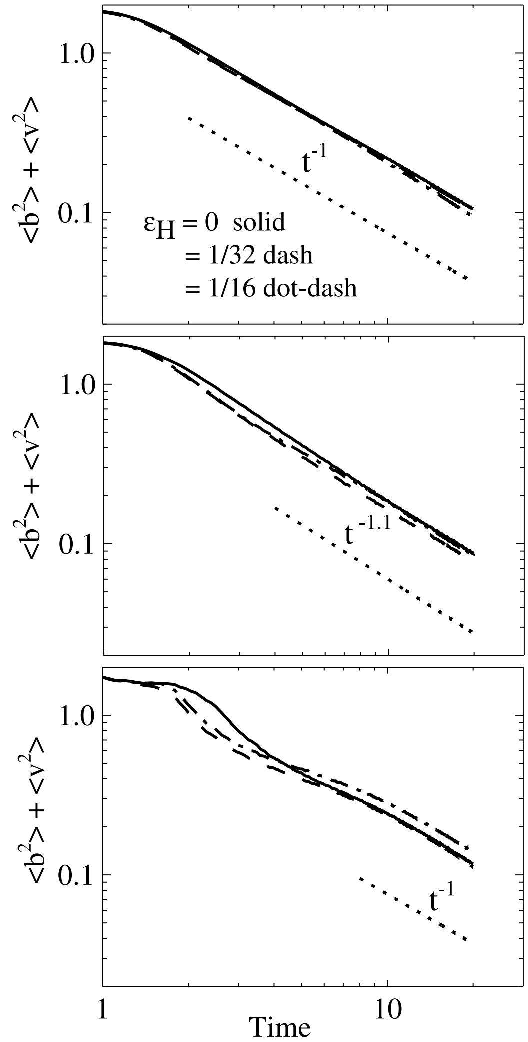

Figure 2. Energy decay for ϵ_H = 0, 1/32 and 1/16 (dot-dash) cases, 128^3 runs with identical initial data, and $B_0 = 0$ (top), B_0 = 1, and B_0 = 8 (bottom).

vorticity, thus comparing small-scale structure in the magnetic and velocity fields. The Hall effect slightly changes the balance between current and vorticity structures. In many astrophysical plasmas this adjustment of small-scale structure may represent the leading-order modification to

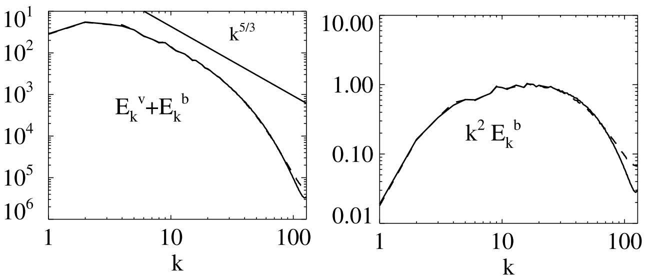

Figure 3. (Left) Energy spectra from 256^3 runs, comparing ϵ_H = 0 and 1/32 ($k^{-5/3}$ shown for comparison.) (Right) Mean square current spectra at same time.

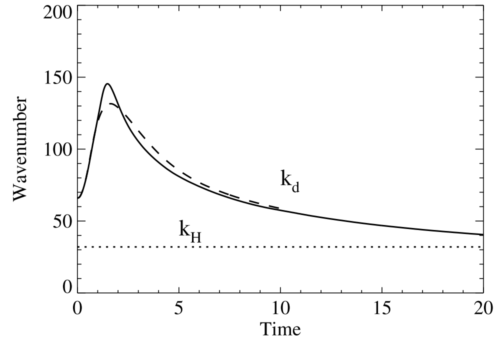

Figure 4. Dissipation wavenumber $k_d = \langle \boldsymbol{\omega}^2 + \mathbf{j}^2 \rangle^{1/4} / \sqrt{\nu}$ vs. time for the Hall 256^3 run (dashed) in Figure 1, and an otherwise identical 128^3 run (solid). Dissipation occurs at scales *smaller* than the Hall scale shown here as $\epsilon_H = {k_H}^{-1}$ = 1/32.

MHD turbulence associated with kinetic effects. Nevertheless, we find little effect on turbulence decay rates.

[13] This conclusion is further evidenced by examining a second series of nine runs, with initial conditions as above, but varying B_0 = 0, 1 and 8 and ϵ_H = 0, 1/32 and 1/16. Since the Hall effect modifies the wave dispersion relationship obtained by linearization about a uniform $\mathbf{B}_0$, one might expect enhanced influence of the Hall effect when $B_0 \neq 0$. This provides a range of dynamical conditions, varying from isotropic "zero frequency" turbulence at the one extreme, over to highly anisotropic dynamics with a strong wave character. Again we find a negative result, shown in Figure 2. Both with and without Hall effect, the turbulent fluctuation energy decays very nearly as $\sim 1/t$. This illustrates our principal result.

[14] This result warrants some careful checking. First we examine the adequacy of spatial resolution. Figure 3 shows energy and mean-square current spectra from the 256^3 simulations (cf. Figure 1). It is evident that the simulations are well-resolved (spectra are well contained in k-space), and there are no significant differences due to Hall effect except at the highest k. Second, the Hall effect engages at scales $\geq$ the dissipation scale, so that either cascade or dissipation processes, or both, may be influenced by the Hall term. To confirm this, we compare the dissipation wavenumber $k_d = \langle \boldsymbol{\omega}^2 + \mathbf{j}^2 \rangle^{1/4} / \sqrt{\nu}$ with the Hall scale $k_H \approx {\epsilon_H}^{-1}$ = 32. Figure 4, shows that $k_d > k_H$ throughout the simulations shown in Figure 1, and for a similar case at lower resolution. It is evident that the dissipation wavenumber changes very little with changing resolution, supporting the conclusion that the simulations are well resolved, and that the Hall effect has had an opportunity to modify the cascade.

[15] An earlier study [*Turner*, 1983] of Hall effects on MHD relaxation considered steady properties of statistical equilibrium, and not the free decay problem. Thus, as far as we are aware there has previously been no clear answer to the question we have raised for low-frequency large-scale MHD energy decay.

[16] The main conclusion we report here—that *the Hall effect in Ohm's law does not significantly affect decay rates of homogeneous turbulence*—has been verified for Hall parameter ranging from ϵ_H = 1/32 to 1, for various initial

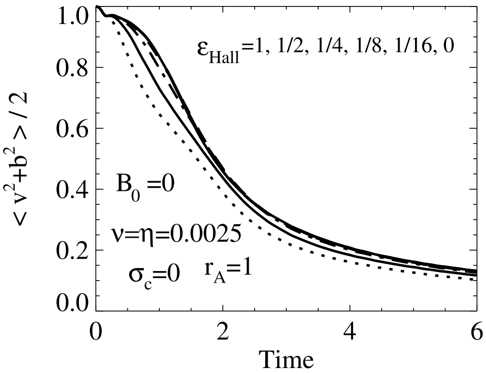

Figure 5. Energy decay in a series of 64^3 runs with $R = R_m = 1/400$, $B_0 = 0$ and varying ϵ_H.

states. Figure 5 shows an additional sampling of such runs, once again the conclusion is that the Kármán-Taylor decay rate is well maintained and there is no significant change associated with the presence of the Hall term even when it is rather strong, and much larger relative to energy scale than in most astrophysical applications.

[17] We also performed runs (not shown) with the ratio $\langle|\mathbf{v}|^2\rangle/\langle|\mathbf{b}|^2\rangle$ initially 1/10 or 10, keeping $\nu = \mu$ and low initial H_c and H_m. Again, the decay law remains approximately $\sim 1/t$, for values of B_0 from 0 to 8. Other runs (also not shown) with initial high H_m but low H_c and $B_0 = 0$, exhibit slower energy decay $\approx t^{-0.6}$ [*Biskamp and Müller*, 2000], but again no significant influence of Hall effect.

[18] These simulations provide a partial answer to the central question at hand. It is difficult to state with certainty that there are no parameters for which the Hall effect would be more pronounced. For example, absolute equilibrium statistical mechanics of Hall MHD [*Turner*, 1983] suggests that large-scale inverse transfer effects should appear when both magnetic and cross helicities are present. This might lead to changes in global energy transfer rates, and we plan future investigation of those cases. However, Gibbs equilibrium studies address neither reconnection nor energy dissipation in the zero helicity limits, so definitive guidance on this issue has been lacking. Based upon numerical experiments, we find for cases with rather widely varying parameters, that the Hall effect does *not* have substantial influence on the global energetics of turbulent decay. Our results suggest that the Hall effect is responsible for a modest rearrangement in the generation of mean-square vorticity and mean-square current, but evidently not the total amount of dissipation. This appears to be consistent with the idea [*Shay et al.*, 1998; *Wang et al.*, 2000] that the Hall effect permits the protons (the momentum bearing component in simple MHD) to decouple from the magnetic field at a scale larger than would be otherwise possible.

[19] A remaining question is whether the present results stand in contradiction to the proposed interpretation of recent (GEM) simulations of collisionless reconnection [*Birn et al.*, 2001; *Shay et al.*, 1998; *Wang et al.*, 2000]. We believe there is no contradiction. In the GEM studies, reconnection commences from an equilibrium, initiated by a small monochromatic perturbation. In such cases, the detailed structure of the reconnection layer influences the rate at which the process proceeds. In turbulent dynamics however, there is always a substantial amount of flow energy available to drive parcels of plasma towards one another. Turbulence is therefore not a "spontaneous" reconnection scenario, but rather is driven. Our results suggest that the details of the kinetic physics in the reconnection layer when driving is present may not be of substantial importance with regard to global energetics.

[20] Understanding how turbulence affects collisionless plasmas remains an exceedingly important topic in space physics and astrophysics. Magnetohydrodynamics has been and is likely to remain a central dynamical description in such studies. However it has become increasingly recognized that modifications of, and limitations to, the MHD description must be understood in order to clearly link MHD results to kinetic plasma descriptions. The present results provide a step in this direction. There are other questions regarding kinetic effects on MHD turbulence, and other ranges of parameters to be explored. In particular, the influence of nonzero cross helicity and magnetic helicity (expected from Gibbs ensemble studies [*Turner*, 1983], and the effect of higher Reynolds numbers (kinetic and magnetic) need to be explored. In addition, the related problem of the interplay of turbulence and Hall effects on large-scale reconnection processes emerges as a priority. We plan to address some of these in future communications.

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