Extension of the Riemann $\xi$-function’s logarithmic derivative positivity region to near the critical strip

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Kevin A. Broughan

Department of Mathematics, University of Waikato, Hamilton, New Zealand
E-mail: kab@waikato.ac.nz

The inequality
\[ \Re \frac{\xi'(\sigma + it)}{\xi(\sigma + it)} > \frac{\xi'(\sigma)}{\xi(\sigma)} \]
for $t \neq 0$ is extended to the region $\sigma \geq 1 + 1/(\log |t| - 5)$ for all $t \neq 0$ and for $\sigma \geq 1$ for $t$ sufficiently large or small.

Key Words: Riemann zeta function, xi function, zeta zeros.

MSC2000: 11M26, 11R42.

1. INTRODUCTION

In the paper [3] Lagarias shows that, assuming the Riemann hypothesis,
\[ \Re \frac{\xi'(\sigma + it)}{\xi(\sigma + it)} > \frac{\xi'(\sigma)}{\xi(\sigma)} \]
for all $\sigma > 1/2$ and for all $t \neq 0$. He also shows that this inequality holds unconditionally in case $\sigma \geq 10$ and remarks that it seems likely the inequality could be established unconditionally for $\sigma > 1 + \epsilon$ for any given fixed positive $\epsilon$ “by a finite computation”. Here we derive the inequality unconditionally up to $\sigma = 1$ for $t$ sufficiently small or large, and for mid-range $t$ to $\sigma \geq 1 + 1/(\log |t| - 5)$. This is Theorem 3.1, proved following 5 lemmas. “Sufficiently small” means up-to a value of $t$ which satisfies $|t| \leq \sqrt{2} - \sqrt{2}\gamma$, where $\gamma$ is the y-coordinate of the first off critical line non-trivial zero of $\zeta(s)$. “Sufficiently large” means greater than $e^{e^{16e^{-13}}}$. 

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where \( c_1 \) is the absolute constant appearing in the inequality for the logarithmic derivative of \( \zeta(s) \).

## 2. PRELIMINARY LEMMAS

**Lemma 2.1.** There exists an absolute constant \( c_1 \) such that for all \( \sigma \geq 1 \) and all \( t \geq t_1 > 0 \)

\[
\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq c_1 (\log t)^{2/3} (\log \log t)^{1/3}.
\]

**Proof.** This follows from Richert [5] or Cheng [1]. See also [6, Section 6.19].

**Lemma 2.2.** For \( \sigma > 1 \) let

\[
f(\sigma) := \frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{\sigma - 1} - \gamma_o.
\]

where \( \gamma_o \) is Euler’s constant. Then there exists a positive absolute constant \( c_2 \) such that

\[-c_2 (\sigma - 1) < f(\sigma) < 0,
\]

and \( c_2 \) can be taken to be \( \gamma_o^2 - 2\gamma_1 \) where

\[
\gamma_1 = -\lim_{N \to \infty} \left( \sum_{m=2}^{N} \frac{\log m}{m} - \frac{\log^2 N}{2} \right) = -0.07235..
\]

so \( c_2 = 0.47789.. \)

**Proof.** Write

\[
f(\sigma) = \frac{1}{\sigma - 1} - \gamma_o - \sum_{p,m \geq 1} \frac{\log p}{p^m \sigma}
\]
so

\[
 f'(\sigma) = \sum_{p,m \geq 1} \frac{m \log^2 p}{pm^2} - \frac{1}{(\sigma - 1)^2} \\
= \sum_p \log^2 p \sum_{m \geq 1} m \left( \frac{1}{(p^\sigma m)^2} \right) - \frac{1}{(\sigma - 1)^2} \\
= \sum_p \log^2 p \sum_{m \geq 1} \left( \frac{1}{(p^\sigma - 1)^2} \right) - \frac{1}{(\sigma - 1)^2}.
\]

Hence

\[
 f''(\sigma) = \frac{2}{(\sigma - 1)^3} + \sum_p \frac{\log^3 p.p^\sigma}{(p^\sigma - 1)^2} - 2 \sum_p \frac{\log^3 p.p^\sigma}{(p^\sigma - 1)^3} \\
= \frac{2}{(\sigma - 1)^3} + \frac{\log^3 2.2^\sigma(2^\sigma - 3)}{(2^\sigma - 1)^3} + \sum_{p \geq 3} \frac{\log^3 p.p^\sigma(p^\sigma - 3)}{(p^\sigma - 1)^3}.
\]

If \( \sigma \geq \log_3 3 \) each term is non-negative so \( f''(\sigma) > 0 \). If \( 1 < \sigma < \log_3 3 \) the sum of the first two terms is positive, so in all cases \( f''(\sigma) > 0 \). Hence \( f(\sigma) \) is concave upwards on \((1, \infty)\).

Now the Laurent expansion of \( \zeta(s) \) in the neighborhood of \( s = 1 \) [2, Theorem 1.4] is

\[
 \zeta(s) = \frac{1}{s - 1} + \gamma_0 + \gamma_1(s - 1) + ...
\]

where, for \( k \geq 0 \)

\[
 \gamma_k = \frac{(-1)^k}{k!} \lim_{n \to \infty} \left( \sum_{m=1}^n \frac{\log^k m}{m} - \frac{\log^{k+1} n}{k+1} \right).
\]

so \( \gamma_0 \) is Euler’s constant and \( \gamma_1 < 0 \). From this it follows that, in a neighborhood of \( s = 1 \),

\[
 \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s - 1} - \gamma_0 = (2\gamma_1 - \gamma_0^2)(s - 1) + O((s - 1)^2)
\]

so \( f'(1) = 2\gamma_1 - \gamma_0^2 \). Therefore, by the concavity of \( f(\sigma) \), \( f(\sigma) > (2\gamma_1 - \gamma_0^2)(\sigma - 1) \) for \( \sigma > 1 \).

Now, by continuous extension, \( f(1) = 0 \) and \( f'(1) < 0 \). If there was a value \( \sigma > 1 \) with \( f(\sigma) \geq 0 \) then, by Rolle’s theorem, there would be a value
with \( f'(\sigma) = 0 \) and so, since \( \lim_{\sigma \to \infty} f(\sigma) = -\gamma_0 \), a point with \( f''(\sigma) = 0 \). But by what we have proved this is impossible. Hence \( f(\sigma) < 0 \) for all \( \sigma > 1 \).

**Lemma 2.3.** If \( 1 \leq \sigma < 10 \) and \( t \geq t_2 > 0 \):

\[
\Re \frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)} \geq \log \frac{t}{2} - 2 - \frac{2}{5t^2}.
\]

**Proof.** This follows directly using the asymptotic expression

\[
\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + R, |R| \leq \frac{1}{10|z|^2}, |z| \geq 2, \Re z > 0.
\]

and the bound

\[
\left| \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)} \right| \leq 2 \]

which holds for \( 1 \leq \sigma \leq 10 \).

**Lemma 2.4.** Let \( \sigma \geq 1, 0 < \beta < 1/2, \gamma > 0 \) be real numbers and define

\[
h(t) := \frac{(\gamma - t)^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + (\gamma - t)^2)((\sigma + \beta - 1)^2 + (\gamma - t)^2)},
\]

and \( f(t) := h(t) + h(-t) \). Let \( c_4 = \sqrt{2 - \sqrt{2}} \). Then for all \( t \) with \( |t| < c_4 \gamma \), \( f(t) > f(0) \).

**Proof.** Define \( u := \sigma - 1/2 \) and \( v := \sigma + \beta - 1 \). Then \( u > v > 0 \) and we can write

\[
h(t) = \frac{(\gamma - t)^2 + uv}{((\gamma - t)^2 + u^2)((\gamma - t)^2 + v^2)}
\]

\[
= \frac{1}{u + v} \left( \frac{u}{(\gamma - t)^2 + u^2} + \frac{v}{(\gamma - t)^2 + v^2} \right).
\]

Then

\[
f(t) = h(t) + h(-t)
\]

\[
= \frac{1}{u(u + v)} \left( \frac{1}{(\gamma - t)^2/u^2 + 1} + \frac{1}{(\gamma + t)^2/u^2 + 1} \right)
\]

\[
+ \frac{1}{v(u + v)} \left( \frac{1}{(\gamma - t)^2/v^2 + 1} + \frac{1}{(\gamma + t)^2/v^2 + 1} \right).
\]
Let
\[ g_\gamma(t) := \frac{1}{(\gamma - t)^2 + 1} + \frac{1}{(\gamma + t)^2 + 1}. \]

Then the derivative
\[ g'_\gamma(t) = \frac{4t(-t^4 - 2(1 + \gamma^2)t^2 + (3\gamma^4 + 2\gamma^2 - 1))}{((\gamma - t)^2 + 1)^2((\gamma + t)^2 + 1)^2} \]
and \( g'_\gamma(0) = 0 \), \( g'_\gamma \) is an odd function of \( t \), and the numerator is positive if
\[ 0 < t < (\gamma^2 + 1)^{1/4}(2\gamma - (\gamma^2 + 1)^{1/2})^{1/2}, \]
or for the slightly smaller but more convenient range \( 0 < t < \sqrt{2 - \sqrt{2\gamma}} = c_4\gamma \), so \( g'_\gamma(t) > 0 \) in this range. Hence
\[ f'(t) = \frac{1}{u^2(u + v)}g'_x \left( \frac{t}{u} \right) + \frac{1}{v^2(u + v)}g'_y \left( \frac{t}{v} \right) \]
is positive for \( t \) with \( 0 < t/u < c_4\gamma/u \) and \( 0 < t/v < c_4\gamma/v \), that is the same range as before. Therefore \( f(0) < f(t) \). But \( f(t) \) is even, so the same inequality holds for \( t \) negative also.

**Lemma 2.5.** Let \( c_0 \) be a positive real number representing the y coordinate of the first zeta zero which is off the critical line (assuming such a zero exists). If \( 0 < t < c_4c_0 \) and \( 1 \leq \sigma < 10 \), then
\[ \Re \xi'(\sigma + it) > \frac{\xi'(\sigma)}{\xi(\sigma)} \]

**Proof.** With the same notation as in Levinson and Montgomery [4], we can write
\[
\Re \frac{\xi'((\sigma + it))}{\xi((\sigma + it))} - \frac{\xi'((\sigma))}{\xi((\sigma))} = (\sigma - 1/2)(I(\sigma, t) - I(\sigma, 0)), \text{ where}
\]

\[
I(\sigma, t) = T_0 + T_1, \text{ where}
\]

\[
T_0 = \sum_{\beta < 1/2} \left( \frac{(\gamma - t)^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + (\gamma - t)^2)((\sigma + \beta - 1)^2 + (\gamma - t)^2)} - \frac{\gamma^2 + (\sigma - 1/2)^2 - (1/2 - \beta)^2}{((\sigma - \beta)^2 + \gamma^2)((\sigma + \beta - 1)^2 + \gamma^2)} \right)
\]

\[
T_1 = \sum_{\beta = 1/2} \left[ \frac{1}{(\sigma - 1/2)^2 + (t - \gamma)^2} - \frac{1}{(\sigma - 1/2)^2 + \gamma^2} \right].
\]

The proof of Lagarias [3], assuming the Riemann Hypothesis, shows that \(T_1 > 0\) whether or not the Riemann hypothesis is assumed to be true. Lemma 2.4 shows that, since each \(\gamma \geq c_0\), each term in the sum is positive for \(t < c_4 c_0\) so the Lemma follows directly. \(\square\)

### 3. PROOF OF THEOREM 3.1

**Theorem 3.1.** Let \(1 \leq \sigma < 10\) and \(t \neq 0\). Then there exist absolute constants \(c_1, c_2\) so that (unconditionally) for \(\{\sigma + it : |t| \leq c_1, 1 \leq \sigma < 10\}\) or \(\{\sigma + it : |t| \geq c_2, 1 \leq \sigma < 10\}\) or \(\{\sigma + it : \log |t| \geq 5 + 1/(\sigma - 1), 1 < \sigma < 10\}\), we have

\[
\Re \frac{\xi'((\sigma + it))}{\xi((\sigma + it))} > \Re \frac{\xi'((\sigma))}{\xi((\sigma))}
\]

**Proof.** Let \(s = \sigma + it\) and \(1 < \sigma\). Then since

\[
\xi(s) = s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2,
\]
we can write
\[
\Delta_0 := \Re\frac{\xi'(s)}{\xi(s)} - \frac{\xi'(\sigma)}{\xi(\sigma)} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, \text{where}
\]
\[
\Delta_1 := -\frac{t^2}{\sigma(\sigma^2 + t^2)},
\]
\[
\Delta_2 := -\frac{t^2}{(\sigma - 1)((\sigma - 1)^2 + t^2)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)},
\]
\[
\Delta_3 := \Re\frac{\zeta'(s)}{\zeta(s)},
\]
\[
\Delta_4 := \frac{1}{2}\left(\frac{\Gamma''(s/2)}{\Gamma(s/2)} - \frac{\Gamma''(\sigma/2)}{\Gamma(\sigma/2)}\right).
\]

Firstly \(\Delta_1 > -1/\sigma\). From Lemma 2.2 it follows that
\[
\Delta_2 \geq -\gamma_0 + \frac{\sigma - 1}{t}.
\]
By Lemma 2.1 we can write
\[
\Delta_3 = \Re\frac{\zeta'(s)}{\zeta(s)} \geq -c_1 (\log t)^{\frac{2}{3}} (\log \log t)^{\frac{1}{3}}.
\]
By Lemma 2.3 we can write
\[
\Delta_4 = \log t - c_3 \theta
\]
for some small positive constant \(c_5\) (we can take \(c_5 = 4\)) and real \(\theta\) with \(|\theta| < 1\).
Hence \(\Delta_0 > 0\) if
\[
\log t - c_1 (\log t)^{2/3} (\log \log t)^{1/3} > 4,
\]
This is true if and only if
\[
1 - c_1 \left(\frac{\log \log t}{\log t}\right)^{\frac{4}{3}} > \frac{4}{\log t}.
\]
If we assume \(t \geq t_4 := e^8\) then \(1 - 4/\log t \geq 1/2\), so with this restriction we require
\[
\frac{\log \log t}{\log t} < \frac{1}{8c_1^4}.
\]
This inequality holds if \( t \geq t_4 := e^{e^{16c_4^2}} \).

So provided \( \gamma_0 c_4 \geq t_4 \) the two regions \((0, \gamma_0 c_4], [t_4, \infty)\) overlap and the Lagarias inequality holds for all \( \sigma > 1 \). If however \( t_4 > \gamma_0 c_4 \) we argue differently. First let

\[
\begin{align*}
\Delta_1' &:= -\frac{t^2}{(\sigma - 1)((\sigma - 1)^2 + t^2)}, \\
\Delta_3' &:= \Re \left( \frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)} \right), \\
\Delta_0 &= \Delta_1 + \Delta_2' + \Delta_3' + \Delta_4. 
\end{align*}
\]

Since \( |\frac{\zeta'(s)}{\zeta(s)}| \leq -\frac{\zeta'(\sigma)}{\zeta(\sigma)} \), \( \Delta_0' \geq 0 \) for all \( t \geq 0 \). Therefore

\[
\Delta_0 > -\frac{1}{\sigma} - \frac{1}{\sigma - 1} + \log t - 4,
\]

so \( \Delta_0 > 0 \) if \( \log t > 4 + 1/\sigma + 1/(\sigma - 1) \) and this is true if \( \log t > 5 + 1/(\sigma - 1) \). The best uniform value of \( \sigma \) which may be obtained using this method is given approximately by

\[
\sigma_0 = 1 + \frac{1}{\log c_4 c_0 - 5}.
\]

If we assume \( c_4 c_0 = 10^8 \) this leads to \( \sigma_0 = 14/13 \). Strengthening of the above approach to the Lagarias problem requires the derivation of a good explicit value for the constant \( c_1 \) (compare [1]) and knowledge of the best current value value for \( c_0 \) (currently \( 3.2 \times 10^9 \)) [7].

REFERENCES

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