Abstract

If $q^\alpha$ is the Euler factor of an odd perfect number $N$, then we prove that its so called index $\sigma(N/q^\alpha)/q^\alpha \geq 3^2 \times 5 \times 7 = 315$. It follows that for any odd perfect number, the ratio of the non-Euler part to the Euler part is greater than $3^2 \times 5 \times 7/2$.

keywords: odd perfect number, index.

MSC2010: 11A25.

1. Introduction

The main motivation for studying the structure of an odd perfect number is ultimately to establish that such a number cannot exist. It is known that any odd perfect number $N$ must have at least 9 distinct prime factors [10], be larger than $10^{1500}$ [12], have a squarefree core which is less than $2N^{3/2}$ [9], and every prime divisor is less than $(3N)^{1/3}$ [1]. These results represent recent progress on what must
be one of the oldest current problems in mathematics.

Following Dris [5], in this paper we define the index \( m \) of a prime power dividing \( N \). Using a lower bound for the index one can derive an upper bound, in terms of \( N \), for the Euler factor of \( N \). Dris found the bound \( m \geq 3 \) then Dris and Luca [6] improved this to \( m \geq 6 \). In [4] a list of forms in terms of products of prime powers, which includes the results of Dris and Dris-Luca, is derived. We improve the method of [4], obtaining an expanded list of prime power products which cannot occur as the value of an index. This enables us to conclude, in the case of the Euler factor, that \( m \geq 315 \); for any other prime, if the Euler factor divides \( N \) to a power at least 2 then \( m \geq 630 \), and if the Euler factor divides \( N \) to the power 1 then \( m \geq 210 \).

Notations: \( \Omega(n) \) is the total number of prime divisors of \( n \) counted with multiplicity, \( \omega(n) \) the number of distinct prime divisors of \( n \), \( \omega_0(n) \) the number of distinct odd prime divisors of \( n \), \( \sigma(n) \) the sum of the divisors of \( n \), \( d(n) \) the number of divisors of \( n \), \( \log_2 n \) the logarithm to base 2, \( (a, b) \) the greatest common divisor, \( p^c \parallel n \) means \( p^c \) divides \( n \) but \( p^{c+1} \) does not, \( \nu_p(n) \) the highest power of \( p \) which divides \( n \), and \( \text{ord}_p a \) the smallest power of \( a \) which is congruent to 1 modulo \( p \). The symbol \( \Box \), when not being used to denote the end of a proof, represents the square of an integer.

Let \( N \) denote an odd perfect number, and \( q \) a prime divisor with \( q^\alpha \parallel N \) say. We write the standard factorization of \( N \) as

\[
N = q^\alpha \times \prod_{i=1}^{k} p_i^{\lambda_i} \times \prod_{j=k+1}^{s} p_j^{\lambda_j}
\]

where for \( 1 \leq i \leq k \) we have

\[
\sigma\left(p_i^{\lambda_i}\right) = m_i q^{\beta_i}, \quad \beta_i \geq 0, \quad (m_i, q) = 1, \quad m_i > 1.
\]  

These prime numbers \( p_i \) are called primes of \textbf{type 1}. For \( k + 1 \leq j \leq s \)

\[
\sigma\left(p_j^{\lambda_j}\right) = q^{\beta_j}, \quad \beta_j > 0
\]  

and the \( p_j \) are called primes of \textbf{type 2}.

One defines the \textbf{index} or \textbf{perfect number index at prime} \( q \) to be the integer

\[
m := \frac{\sigma\left(N/q^\alpha\right)}{q^\alpha};
\]

in particular \( m = m_1 \cdots m_k \).
In fact $4 \nmid m$, $q \nmid m$, and if an odd prime $p$ satisfies $p^r \mid m$ then $p^r \mid N$. Furthermore if $q$ is the Euler prime, then $m$ is odd and each $m$ corresponding to any other prime is even. Lastly we have the fundamental equation

$$m \times \sigma(q^\alpha) = 2 \times \prod_{i=1}^k p_i^{\lambda_i} \times \prod_{j=k+1}^s p_j^{\lambda_j} = \frac{2N}{q^\alpha}. \tag{4}$$

2. Preliminary Results

First we state the theorem of Chen and Chen [4].

**Theorem 1.** If $N$ is an odd perfect number with a prime power $q^\alpha \parallel N$, then the index $m := \sigma(N/q^\alpha)/q^\alpha$ is not equal to any of the six forms

$$\{p_1, p_1^2, p_1^3, p_1^4, p_1p_2, p_2^2p_2\}
$$

where $p_1$ and $p_2$ are any distinct primes.

The following lemma comes from [6]. Here we give an alternative proof.

**Lemma 2.** If for some $j$ with $k + 1 \leq j \leq s$ (so $p_j$ is a prime of type 2) and for some $\gamma$ with $2 \leq \gamma \leq \lambda_j$ we have $p_j^{\gamma} \mid (q^{\alpha+1} - 1)/(q - 1)$, then $p_j^{\gamma-1} \mid \alpha + 1$.

**Proof.** Because $p_j^{\gamma}(1 + p_j + \cdots + p_j^{\lambda_j - 1}) = q^{\gamma} - 1$ one deduces $p_j^{\gamma} \mid q^{\gamma} - 1$, in which case $p_j^{\gamma} \mid q^{\ord p_j(q)} - 1$. However

$$2 \leq \gamma \leq \nu_{p_j}\left(\frac{q^{\alpha+1} - 1}{q - 1}\right) = \nu_{p_j}\left(\frac{q^{\ord p_j(q)} - 1}{q - 1}\right) + \nu_{p_j}\left(\frac{\alpha + 1}{\ord p_j(q)}\right). \tag{5}$$

If $\ord p_j(q) = 1$ then $\gamma \leq \nu_{p_j}(\alpha + 1)$ and $p_j^{\gamma} \mid \alpha + 1$, whereas if $\ord p_j(q) > 1$ one has $\gamma \leq 1 + \nu_{p_j}(\alpha + 1)$ and therefore $p_j^{\gamma-1} \mid \alpha + 1$. \qed

**Lemma 3.** (Ljunggren, see [7]) The only integer solutions $(x, n, y)$ with $|x| > 1$, $n > 2$, $y > 0$ to the equation $(x^n - 1)/(x - 1) = y^2$ are $(7, 4, 20)$ and $(3, 5, 11)$, i.e. $(7^4 - 1)/(7 - 1) = 20^2$ and $(3^5 - 1)/(3 - 1) = 11^2$.

**Lemma 4.** [7] The only solutions in non-zero integers with $n > 1$ to the equation $y^n = x^2 + x + 1$ are $n = 3$, $y = 7$ and $x = 18$ or $x = -19$. 
The following well known result [2, 3, 13] guarantees the existence of primitive prime divisors for expressions of the form \(a^n - 1\) with fixed \(a > 1\).

**Lemma 5.** Let \(a\) and \(n\) be integers greater than 1. Then there exists a prime \(p\) dividing \(a^n - 1\) which does not divide any of \(a^m - 1\) for each \(m \in \{1, \ldots, n-1\}\), except possibly in the two cases \(n = 2\) and \(a = 2^{\beta} - 1\) for some \(\beta \geq 2\), or \(n = 6\) and \(a = 2\). Such a prime is called a **primitive prime factor**.

We complete this set of preliminary results by filling in the missing case from the proof of the fundamental lemma [4, Lemma 2.4].

**Lemma 6.** Let \(N\) be an odd perfect number. Then \(d(\alpha + 1) \leq \omega(N)\) whenever a prime power \(q^\alpha \mid N\).

**Proof.** Let \(n_1, n_2, \ldots, n_w\) denote all the distinct positive divisors of \(\alpha + 1\) which are greater than 1.

If \(2 \mid \alpha + 1\) then \(\alpha\) is odd, and thus \(q \equiv \alpha \equiv 1 \mod 4\). Therefore \(q\) cannot be of the form \(2^3 - 1\) and must be odd. By Lemma 5 there exists a primitive prime factor \(q_i \mid q^{n_i} - 1\); since \(2 \mid q^1 - 1\) the \(q_i\) are all odd, and as they are primitive, one finds \(q_i \nmid q^1 - 1\) also. Hence

\[
q_i \mid \frac{q^{n_i} - 1}{q - 1} \mid \frac{q^{\alpha+1} - 1}{q - 1}
\]

so that \(q_n_1 \cdots q_n_w \mid (q^{\alpha+1} - 1)/(q - 1)\). But \(m \times \sigma(q^\alpha) = 2N/q^\alpha\) thus, including the divisor 1 and recalling \(2 \mid \sigma(q^\alpha)\), one obtains the inequalities

\[
d(\alpha + 1) = w + 1 \leq \omega(\sigma(q^\alpha)) \leq \omega(m \sigma(q^\alpha)) = \omega\left(\frac{2N}{q^\alpha}\right) = \omega(N).
\]

Alternatively if \(2 \nmid \alpha + 1\) then \(\alpha\) is even so, again by Lemma 5, we obtain distinct odd primes \(q_{n_i}\) with

\[
q_{n_1} \cdots q_{n_w} \mid \frac{q^{\alpha+1} - 1}{q - 1}.
\]

Because in this case \(2 \nmid m\) and \(2 \nmid \sigma(q^\alpha)\), we deduce that

\[
d(\alpha + 1) = 1 + w \leq 1 + \omega(\sigma(q^\alpha)) \leq \omega(m \sigma(q^\alpha)) = \omega\left(\frac{2N}{q^\alpha}\right) = \omega(N)
\]

which completes the proof of the lemma. \(\square\)
3. The Proof

We now amend the proof of Theorem 1.1 of [4].

Lemma 7. Let $N$ be an odd perfect number, and $m$ the index at some prime divisor of $N$. Then

$$\Omega(m) + \omega_0(m) \geq \omega(N) - \log_2 \sqrt{\omega(N)} - \eta$$

where $\eta = 1$ if $m$ is odd, $\eta = \frac{1}{2}$ if $m$ is even and the Euler prime divides $N$ to a power greater than 1, and $\eta = \frac{3}{2}$ if $m$ is even and the Euler prime divides $N$ exactly to the power 1.

Proof. Whenever $(m, p_{k+1} \cdots p_s) = p_{k+1} \cdots p_s$, one has an inequality

$$s - k \leq \omega_0(m) = t$$

and it follows that

$$k + t \geq s = \omega(N) - 1.$$  

Because $k \leq \Omega(m)$, $t = \omega_0(m)$ and $\omega(N) \geq 9$, we quickly deduce

$$\Omega(m) + \omega_0(m) \geq k + t \geq \omega(N) - 2 \geq \omega(N) - \log_2 \sqrt{\omega(N)} - 0.42.$$  

The non-trivial case occurs when $(m, p_{k+1} \cdots p_s) \neq p_{k+1} \cdots p_s$. By suitably reordering the $p_i$, we can always write for some $l$ with $k \leq l < s$:

$$\frac{p_{k+1} \cdots p_s}{(m, p_{k+1} \cdots p_s)} = p_{l+1} \cdots p_s.$$  

Applying [4] Equation (2.2) and (6), we see that

$$p_{l+1}^{\lambda_i} \cdots p_s^{\lambda_s} \mid \sigma(q^\alpha).$$

Moreover using [4] Equation (2.1) and [4] Lemma 2.3,

$$p_i^{\lambda_i - 1} \mid \alpha + 1, \quad l + 1 \leq i \leq s$$

hence

$$p_{l+1}^{\lambda_{l+1} - 1} \cdots p_s^{\lambda_s - 1} \mid \alpha + 1.$$  

Now for $k + 1 \leq i \leq s$ one knows $\sigma(p_i^{\lambda_i}) = q^{\beta_i}$, and $q$ is odd so we must have $\lambda_i$ even. It follows for $l + 1 \leq i \leq s$ each $\lambda_i \geq 2$, thus $p_{l+1} \cdots p_s \mid \alpha + 1$. Note also that $l < s$ in which case $s - l \geq 1$. 


If \( s - l = 1 \) then because \( \omega(N) \geq 9 \),

\[
\Omega(m) + \omega_0(m) \geq k + t \geq l = s - 1 \geq \omega(N) - 2 \geq \omega(N) - \log_2 \sqrt{\omega(N)} - 0.42
\]
as in the previous case.

If \( s - l \geq 2 \) then we claim at most one of the \( \lambda_i = 2 \) and the remainder have \( \lambda_i \geq 4 \).

To see this, consider the equations

\[
p_{i}^2 + p_i + 1 = q\beta_i.
\]

If \( \beta_i > 1 \) then, by Lemma 4, the only solution is \( \beta_i = 3, q = 7 \) and \( p_i = 18 \) which is not prime, so the solution cannot occur in this context. Hence \( \beta_i = 1 \) and the form of the equation is \( q = x^2 + x + 1 \). But this, for given \( q \), has at most one positive integer solution, therefore at most one prime solution \( p_i \).

By renumbering the \( p_i \) if necessary, when \( s - l \geq 2 \) we can write

\[
\frac{3}{3} p_{i+1}^3 p_{i+2}^3 \cdots p_{s-1}^3 p_s | \alpha + 1.
\]

**Case 1:** Suppose that the index \( m \) is odd. Then \( q \) is the Euler prime, and consequently \( 2 | \alpha + 1 \). Hence

\[
2^{2s-2l} \leq d(\alpha + 1) \leq \omega(N),
\]
or in other words \( s - l \leq \log_2 \sqrt{\omega(N)} \), which implies

\[
l \geq \omega(N) - \log_2 \sqrt{\omega(N)} - 1.
\]

As \( \omega_0(m) = t \) then by Equation (6) we have \( l - k \leq t \), so \( l \leq \Omega(m) + \omega_0(m) \). Lastly because \( \omega(N) \geq 9 \),

\[
6.41 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - 1 \leq l \leq \Omega(m) + \omega_0(m).
\]

**Case 2:** Here we assume the Euler prime divides \( N \) to a power at least 2. Let \( m \) be even. Now \( m = m_1 \cdots m_k \) and \( 2 | m \) so, for a unique \( i \), one knows that \( 2 | m_i \). We claim that \( 2 \neq m_i \). If not, then

\[
\sigma(p_i^{\lambda_i}) = 2q^\beta_i
\]
whence \( p_i \) is the Euler prime and \( \lambda_i + 1 \) is even; we can write
\[
\frac{p_i^{\lambda_i + 1} - 1}{2(p_i - 1)} = \left( \frac{p_i^{\lambda_i + 1}}{p_i - 1} \right) x \left( \frac{p_i^{\lambda_i + 1}}{2} + 1 \right) = q_i^{\beta_i},
\]
but this cannot hold since the two factors in the middle term are coprime and greater than 1, thus \( 2 \neq m_i \).

It follows that \( k \leq \Omega(m) - 1 \). In this scenario with \( s - l \geq 2 \), we also know
\[
p_i^3 p_{i+1}^3 \cdots p_{s-1}^3 \mid \alpha + 1
\]
thus
\[
2^{2s - 2l - 1} \leq d(\alpha + 1) \leq \omega(N),
\]
which in turn implies
\[
l \geq s - \frac{1}{2} - \log_2 \sqrt{\omega(N)} = \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2}.
\]
It follows from the discussion that
\[
\omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2} \leq l \leq k + t \leq \Omega(m) - 1 + \omega_0(m)
\]
and therefore
\[
6.91 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{1}{2} \leq \Omega(m) + \omega_0(m).
\]

**Case 3:** We shall now assume the Euler prime divides \( N \) exactly to the power 1 and that \( m \) is even. Here we have only the weaker inequality \( k \leq \Omega(m) \), and using an identical argument to **Case 2**:
\[
5.91 \leq \omega(N) - \log_2 \sqrt{\omega(N)} - \frac{3}{2} \leq \Omega(m) + \omega_0(m).
\]

\( \square \)

**Lemma 8.** If the index \( m \) is a square, then \( \alpha = 1 \).

**Proof.** If \( m = \square \) then necessarily \( q \) is the Euler prime. We must have \( \sigma(q^\alpha) = 2\square \) and \( \alpha \) is odd. Assuming \( \alpha > 1 \) then
\[
\frac{1}{2} \left( \frac{q^{\alpha + 1} - 1}{q - 1} \right) = \left( \frac{q^{(\alpha + 1)/2} - 1}{q - 1} \right) x \left( \frac{q^{(\alpha + 1)/2} + 1}{2} \right) = \square
\]
and the two factors in the penultimate term are coprime, in which case
\[ \frac{q^{(\alpha+1)/2} - 1}{q - 1} = \Box. \]

By Lemma 3 one has \((\alpha+1)/2 \leq 2\), and as \(\alpha \equiv 1 \mod 4\) we deduce \(\alpha = 1\), thereby yielding a contradiction. \(\square\)

**Lemma 9.** If the index \(m\) is odd, then it cannot be the sixth power of a prime.

**Proof.** Firstly the index being odd means it corresponds to the Euler prime. Assume \(m = p^6 = \Box\). By Lemma 8, we have \(\alpha = 1\). If \(p = p_I\) is of type 1 then \(\sigma(p_I^{\lambda_I}) = p^\theta q^{\beta_I}\) for some \(\theta > 0\), which is false. Hence \(p_I\) will be of type 2. If any other prime \(p_j\) were also of type 2, then due to the equality
\[ \sigma(q^\alpha) = \frac{2N}{q^\beta p_I^\lambda} \]
we would have \(p_j^2 \mid \sigma(q^\alpha)\) and also \(q^{\beta_j} = p_I^{\lambda_j+1} - 1\); however from Lemma 2 there is a divisibility \(p_j \mid \alpha + 1 = 2\), which is clearly false as \(p_j \geq 3\).

Consequently there exists exactly one type 2 prime, \(p_I\). Note that \(\lambda_I \geq 6\). If \(\lambda_I \neq 6\) we would have \(\lambda_I\) even and greater than 6, implying \(p_I^2 \mid \sigma(q^\alpha)\) and by Lemma 2, \(p_I \mid \alpha + 1\) which is false. Hence \(\lambda_I = 6\) and we can write
\[ \sigma(q^\alpha) = 2p_I^{\lambda_I} \cdots p_k^{\lambda_k}. \]
But \(m = p^6 = m_1 \cdots m_k\) has at most 6 factors, in which case \(k \leq 6\); therefore
\[ 9 \leq \omega(N) = k + 2 \leq 8 \]
a clear contradiction, completing the proof that \(m \neq p^6\). \(\square\)

Applying Lemmas 7 and 9, we have shown

**Theorem 10.** If \(N\) is an odd perfect number and the odd prime \(q^\alpha \parallel N\) then the index \(\sigma(N/q^\alpha)/q^\alpha\) is either odd when \(q\) is the Euler prime, or even but not divisible by 4 when \(q\) is not the Euler prime.

(i) If \(q\) is the Euler prime, it cannot take any of the 11 forms
\[ \{p, p^2, p^3, p^4, p^5, p^6, p_1 p_2, p_1^2 p_2, p_1^3 p_2, p_1^2 p_2^2, p_1 p_2 p_3\} \]
where \( p \) is any odd prime and \( p_1, p_2, p_3 \) are any distinct odd primes.

(ii) If \( q \) is not the Euler prime and the Euler prime divides \( N \) to a power greater than 1, it cannot take any of the 7 forms

\[ \{2, 2p, 2p^2, 2p^3, 2p_1p_2, 2p_1^2p_2\}. \]

(iii) If \( q \) is not the Euler prime and the Euler prime divides \( N \) to the power 1, it cannot take any of the 5 forms

\[ \{2, 2p, 2p^2, 2p^3, 2p_1p_2\}. \]

Therefore the smallest possible value of the index \( m \) is, respectively:

\[ 3^2 \times 5 \times 7 = 315 \] in case (i),

\[ 2 \times 3^3 \times 5 = 270 \] in case (ii),

and \[ 2 \times 3^2 \times 5 = 90 \] in case (iii).

**Corollary 11.** It follows directly that for any odd perfect number, the ratio of the non-Euler part to the Euler part is greater than 315/2.

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**References**


