THE STRUCTURE OF SECTORS OF ZEROS OF ENTIRE FLOWS

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Abstract. Dynamical systems or flows \( \dot{z} = f(z) \), where \( f \) is entire on \( \mathbb{C} \), are considered. The nature of the separatrices, the structure of sectors and the boundaries of sectors of the flow at the zeros of \( f(z) \) are determined.

1. Introduction

This paper continues the study, commenced in [1, 2, 3, 9] and continued in [4], to explore the local and global properties of complex functions, especially those useful in number theory, using topological methods based on dynamical systems. This requires an investigation of the flow

\[
\dot{z} = \frac{dz}{dt} = f(z), \quad z \in \Omega
\]

where \( f \) is a complex valued function of a complex variable, \( t \) is a real parameter and \( \Omega \) a non-empty open subset of \( \mathbb{C} \).

When \( f(z) \) is required to be holomorphic there are strong implications for the topology of the flow which results. Some of these were detailed in [3]. For example there are no saddle points. There are no limit cycles on simply connected subdomains. Here this work is taken further.

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Potential applications include a deeper understanding of the zeros of the Riemann zeta function. It is already known [4] that each simple zero on the critical line is a focus. From the results in this paper it follows that each sector at each zero, simple or otherwise, is unbounded.

In Section 2 the term separatrix is defined, as is transit time. In the main Section 3, additional to what was shown in [3], it is proved that the neighborhood of a center is simply connected and that the sum of the transit times for boundary orbits is less than or equal to the period at the center. Each elliptic sector at a higher order zero has a very similar structure, i.e. it is simply connected and the boundary consists of disjoint orbits. The zero itself is the only zero of the flow on the boundary of a sector. Where the zero is a node or focus the picture is somewhat more complicated: zeros on the boundary are possible, but only as part of “separatrix cycles” with at most one or two orbits and a maximum of one zero. In all cases (center, node, focus, elliptic sector) the neighborhood is unbounded. This explains a major feature of the phase portrait of $\zeta(s)$, as may be observed in [4, Figure 1].

There is an occasional need to embed flows onto the Riemann sphere, but the structure of the point at infinity is not used. Implicitly, the flow is replaced by

$$\dot{z} = \frac{f(z)}{1 + |z|^2 + |f(z)|^2}$$

which is real analytic on $\mathbb{C}$ and continuous on $\mathbb{C} \cup \{\infty\}$ when $f$ is holomorphic, has a zero at infinity, and the same phase portrait at $\dot{z} = f(z)$.

2. Definition of Separatrix

In the examples of meromorphic flows considered in [3, 4] the most interesting features have been the zeros and the separatrices. The former are specified very completely in the literature but not the latter. For example if there is a definition of separatrix in [6] or [7] it is hard to find. The definitions which do occur (see below) are adequate for polynomial or rational function flows, but not for more general holomorphic flows. For example [10, page 290] defines a separatrix to be either a critical point, limit cycle or trajectory on the boundary of a hyperbolic sector at a critical point. This does
not take account of separatrices with endpoints “at infinity” or at a pole. The definition in [8, page 223] is better: “a path which is either a limit cycle or a path terminating or beginning on the projective plane with a side of a hyperbolic sector”. However, in the examples of special interest, e.g. the exponential, hyperbolic, gamma, Riemann zeta and Riemann xi functions, behaviour at infinity is far from regular, but each of these functions exhibits strong separatrix phenomena.

The definition below avoids the use of points at infinity and covers the separatrices which have appeared in the examples. Limit cycles do not occur so are not part of the definition. Zeros have their own classification, so have been left out also.

**Definition 2.1** (positive/negative separatrix). We say the orbit $\gamma$ is a **positive separatrix** if for some $z \in \gamma$ the maximum interval of existence of the path commencing at $z$ and proceeding in positive time is finite. We say the orbit $\gamma$ is a **negative separatrix** if for some $z \in \gamma$ the maximum interval of existence of the path commencing at $z$ and proceeding in negative time is finite. The orbit $\gamma$ is a **separatrix** if it is a positive or negative separatrix.

This definition works for functions like $e^z, \zeta(z), \xi(z)$, polynomials, rational functions and the like. For the exponential flow (Figure 1), the separatrices are the lines through $y = n\pi i, n \in \mathbb{Z}$ running parallel to the x-axis. For $ze^z$ there is a node at 0 and the separatrix through $y = 0$ is lost, with the remaining separatrices of $e^z$ being distorted. For $f(z) = 1/z$ the x and y axes, make up four separatrices, consistent with the given definition - see Figure 2.

With this definition of separatrix, the union of all separatrices and zeros remains closed, as it does under the definition of Marcus [9].

**Definition 2.2** (transit time). Let $\dot{z} = f(z)$ be a meromorphic flow and $\gamma$ an orbit. If $a, b \in \gamma$ we define the **transit time** from $a$ to $b$, denoted $\tau(a, b)$, to be the value of the integral

$$\tau(a, b) = \int_a^b \frac{dz}{f(z)}$$

where the integral is evaluated along the path $\gamma$. Note that any continuous deformation of this path will give the same value of the
Figure 1. Exponential function flow: $\dot{z} = e^z$

Figure 2. Flow with one zero: $\dot{z} = ze^z$
integral provided it does not cross a zero of $f(z)$. Note also that if $a = \gamma(t_1)$ and $b = \gamma(t_2)$ then $\tau(a, b) = t_2 - t_1$.

If $\gamma$ is an orbit we define the **transit time** of $\gamma$, denoted $\tau(\gamma)$, to be the length of the maximum interval of existence for the flow commencing at any $z \in \gamma$, if this is bounded above and below, otherwise let $\tau(\gamma) = \infty$.

Transit time is simply the time it takes to go from one point to another on an orbit. It is not defined for points which are on different orbits.

**Definition 2.3.** Let $\alpha$ and $\beta$ be two orbits in the same elliptic sector at a zero (periodic orbits about the same zero). We say $\beta$ is **outside** $\alpha$ if every path from a point in the interior of $\alpha \cup \{z_0\}$ (respectively $\alpha$) to a point on $\beta$ cuts $\alpha$.

**Lemma 2.4.** Let $\dot{z} = f(z)$ be an entire flow. Let $C$ be the graph of an orbit on the boundary of a center or elliptic sector at $z_0$. Let $x, y \in C$ and let $\epsilon > 0$ be given. Then there is an orbit $\gamma$ about $z_0$ such that for all orbits $\beta$ about $z_0$, with $\beta$ outside $\gamma$, $\beta \cap B_\epsilon(x) \neq \emptyset$ and $\beta \cap B_\epsilon(y) \neq \emptyset$.

**Proof.** There exists an orbit $\gamma_1$ which meets $B_\epsilon(x)$ and an orbit $\gamma_2$ which meets $B_\epsilon(y)$. If $\gamma$ is the orbit of the pair $\gamma_1, \gamma_2$ which is outside the other, then $\gamma$ meets both $B_\epsilon(x)$ and $B_\epsilon(y)$: if $\gamma = \gamma_1$ and $z_0$ is a center and $\gamma$ did not meet $B_\epsilon(y)$ then the two open sets consisting of the interior region of $\gamma$ and the exterior region of $\gamma$ would disconnect the connected set $C \cup B_\epsilon(y) \cup \gamma_2$. The case where $C$ is on the boundary of an elliptic sector is similar. \qed

3. **Main Results**

In this section the neighbourhoods of a center (begun in [3, Theorem 3.3]), elliptic sector at a zero, and focus or node of an entire flow $\dot{z} = f(z)$ are described in separate theorems.

**Theorem 3.1** (neighbourhood of a center). Let $z_0$ be a center for the entire flow $\dot{z} = f(z)$ with open neighbourhood $P$ and boundary components $(C_\lambda, \lambda \in \Lambda)$. Then $P$ is simply connected and each $C_\lambda$ is a separatrix. The sum of the transit times of the $C_\lambda$ is bounded by the (common) period of the orbits which circulate about $z_0$. 
Proof. 1. \( P \) is simply connected: If not there exists a closed path \( \Gamma \) in \( P \) such that the interior of \( \Gamma \) has a non-empty intersection with the complement of \( P \). Call this intersection \( A \). Then \( A \) is closed and compact and has a boundary consisting of orbits and zeros of the flow. (These are finite in number since they are contained in a compact subset of \( \mathbb{C} \).) Since there are no limit cycles there must be at least one zero on the boundary. But this is impossible, since the zero would have a hyperbolic sector (from the orbits of \( P \)).

2. Let \( T \) be the period, \( T = 2\pi i / f'(z_0) \) [3, Theorem 2.3]. Then
   \[
   \sum_{\lambda \in \Lambda} \tau(C_\lambda) \leq T:
   \]

Claim A: For all \( \epsilon > 0 \) and \( x, y \in C_\lambda \), there exists a \( \delta > 0 \) such that for all orbits \( \gamma \) with
   \[
   x' \in \gamma \cup B_\delta(x) \neq \emptyset, y' \in \gamma \cup B_\delta(y) \neq \emptyset
   \]
such that \( |\tau(x', y') - \tau(x, y)| < \epsilon \): Chose \( \delta > 0 \) such that, for some constant \( M > 0 \),
   \[
   M \leq |f(z)| \text{ on } B_\delta(x) \text{ and on } B_\delta(y).
   \]
   Then
   \[
   |\tau(x', y') - \tau(x, y)| = \left| \int_{x'}^{y'} \frac{dz}{f(z)} - \int_{x}^{y} \frac{dz}{f(z)} \right| \leq 2\delta 
   \]
   \[
   < \frac{2\delta}{M} \epsilon < \epsilon.
   \]

Claim B: If \( x \in C_\lambda \) and \( \epsilon > 0 \) are given and \( \alpha \cap B_\epsilon(x) \neq \emptyset \)
   and \( \beta \) is outside \( \alpha \), then \( \beta \cap B_\epsilon(x) \neq \emptyset \): the proof of this is similar to
   that of Lemma 2.4.

By [3, Theorem 3.3] the index set is countable (or finite) so we identify it with \( \mathbb{N} \).

Now, for each \( i \in \mathbb{N} \) let \( x_i, y_i \in C_i \) be arbitrary distinct points with the positive direction of flow being from \( x_i \) to \( y_i \). Choose \( \epsilon_1 > 0 \) such that \( B_{\epsilon_1}(x) \cap B_{\epsilon_2}(y) = \emptyset \) and, inductively, \( \epsilon_i > 0 \) such that the entire set of \( B_{\epsilon_i}(x_i) \) and \( B_{\epsilon_i}(y_i) \) are disjoint.

By Claim A, for each \( i \) we can find a \( \delta_i > 0 \) such that any orbit \( \gamma_i \) with
   \[
   x_i' \in B_{\delta_i}(x_i) \cap \gamma_i, y_i' \in B_{\delta_i}(y_i) \cap \gamma_i
   \]
satisfies
   \[
   |\tau(x_i', y_i') - \tau(x_i, y_i)| < \frac{\epsilon}{2i}.
   \]
Then for each $N \in \mathbb{N}$ one of the orbits $(\gamma_i : 1 \leq i \leq N)$ is outside all of the others in the set, and, by Claim B and Claim A, satisfies
\[
\sum_{i=1}^{N} \tau(x'_i, y'_i) \leq T.
\]

But this implies
\[
\sum_{i=1}^{N} \tau(x_i, y_i) \leq T + \epsilon.
\]

Since the points $x_i$ and $y_i$, and $N$ are arbitrary, this implies
\[
\sum_{i=1}^{\infty} \tau(C_i) \leq T + \epsilon,
\]
and the result follows.

3. Each $C_\lambda$ is a separatrix: this is immediate since each $\tau(C_\lambda)$ is bounded. $\square$

Figure 3 represents the flow of a function with 7 zeros, all of them simple:
\[
\dot{z} = z(z^2 + 1)(z^2 - 1)(z^2 - (i + 1)^2),
\]
There are centers at 0 and \( i + 1 \). The boundary of the neighbourhood of 0 consists of two disjoint separatrices, whereas that of \( i + 1 \) has one. Figure 2 of [3] is an example with 4 disjoint separatrices on the boundary of a center. Figure 1 of [3] is an example of an elliptic sector, the next type to be considered.

**Theorem 3.2** (structure of an elliptic sector). Let \( \dot{z} = f(z) \) be an entire flow with a zero at \( z_o \) having order \( n \geq 2 \). Let \( P \) be the set consisting of the union of all of the orbits of the flow in a given sector at \( z_o \) which satisfy \( L_\alpha(\gamma) = L_\omega(\gamma) = z_o \). Then \( P \) is a simply connected open subset of \( \mathbb{C} \) and \( \partial P \) consists of an at most countable union of a set of closed separatrices \( \{ \gamma(x_\lambda, t) : \lambda \in \Lambda, t \in D_\lambda \} \), \( D_\lambda \) being the maximum interval of existence of the flow through \( x_\lambda \), and where each \( \gamma(x_\lambda, t) \) has an unbounded graph, together with two unbounded separatrices, \( u, v \) which satisfy \( L_\omega(u) = z_o = L_\alpha(v) \).

**Proof.**

1. \( P \) is open: If \( z \in P \) then any orbit through a point sufficiently close to \( z \), by continuous dependence on initial conditions, comes arbitrarily close to \( z_o \) in both positive and negative time. Therefore it must tend to \( z_o \) since \( z_o \) is elliptic. Hence \( P \) is open.

2. \( P \) is connected: If \( G, H \) is a disconnecting partition of \( P \) then necessarily \( z_o \in G \) or \( z_o \in H \) and not both. But this is a contradiction since points in either set must come arbitrarily close to \( z_o \) since each must contain complete orbits.

3. \( P \) is simply connected: If not there exists a closed path \( \Gamma \) in \( P \) such that the interior of \( \Gamma \) has a non-empty intersection with the complement of \( P \). Call this intersection \( A \). Then \( A \) is closed and compact and has a boundary consisting of orbits and zeros of the flow. Since there are no limit cycles there must be at least one zero on the boundary. But this is impossible, since the zero would have a hyperbolic sector (from the orbits of \( P \)).

4. \( \partial P = B \) is closed. If \( B = \emptyset \) the proof is complete. Otherwise proceed as follows:

5. Necessarily \( B \) consists of disjoint orbits and zeros. Any zero, other than \( z_o \), would constitute a critical point with a hyperbolic sector, so cannot occur. Chose one point \( x_\lambda \) for each orbit. Let \( D_\lambda = (\alpha, \beta) \) be the maximal interval of existence of \( \gamma(x_\lambda, t) \). Let

\[
B_\lambda = \{ \gamma(x_\lambda, t) \mid t \in D_\lambda \}
\]
Then
\[ B = \{ z_0 \} \cup_{\lambda \in \Lambda} B_\lambda \]
the union being disjoint.

6. Consider \( t \to \beta - \). The argument for \( t \to \alpha + \) is similar. If the image of \([0, \beta]\) is bounded in \( \mathbb{C} \) then necessarily \( \beta = \infty \) and, since the flow has no limit cycle, \( \omega(\gamma) = x_1 \) which would be a critical point with a hyperbolic sector, impossible for a holomorphic flow. Therefore \( B_\lambda \) is unbounded and
\[ B = \bigcup_{\lambda \in \Lambda} \gamma(x_\lambda, t) | t \in D_\lambda = \bigcup_{\lambda \in \Lambda} B_\lambda. \]

7. Each \( B_\lambda \) is closed: If not there is an \( x \in \omega(B_\lambda) \) or \( x \in \alpha(B_\lambda) \). Since the flow has no limit cycle, \( x \) must be a critical point, so must be center, focus, node or point with only elliptic sectors. Since \( B \) is closed, \( x \in B \), so it must have at least one hyperbolic sector, which is false.

8. Each \( B_\lambda \) is a separatrix: by choosing \( r > 0 \) sufficiently small and integrating about an arc of a circle center \( z_0 \) radius \( r \) it follows that, for any two points \( a, b \) on an orbit \( \gamma \) at \( z_0 \) and on a circle center \( z_0 \) radius \( r \):
\[ |\tau(a, b)| \leq \frac{4}{r^{n-1}}. \]
Call this upper bound \( M_r \). Then if \( x, y \in B_\lambda \) are any two points and \( n \in \mathbb{N} \) is given, by the Lemma 2.4, there is an orbit \( \gamma_n \) at \( z_0 \) such that \( B_\lambda \cap \gamma_n \neq \emptyset \) and \( B_\lambda \cap \gamma \neq \emptyset \). If \( x_n \) is in this first intersection and \( y_n \) in the second, then \( |\tau(x_n, y_n)| \leq M_r \). Taking the limit as \( n \to \infty \) shows that \( |\tau(x, y)| \leq M_r \). Since this is true for any pair \( x, y \in B_\lambda \), we have \( \tau(B_\lambda) \leq M_r \), so \( B_\lambda \) is a positive and negative separatrix.

9. The orbits \( u, v \) define the sector. Their behaviour as \( t \to \pm \infty \) can be deduced in a similar manner to those of other orbits on the boundary of \( P \).

10. \( |\Lambda| \leq \aleph_0 \): On the Riemann sphere the \( B_\lambda \) enclose open disjoint regions, which therefore must be at most countable in number.

\[ \square \]

**Theorem 3.3** (structure of a node or focus basin). Let \( \dot{z} = f(z) \) be an entire flow with a simple zero of at \( z_0 \) which is a node or a focus. Let \( P \) be the set of all points in \( \mathbb{C} \) with orbits which tend
to $z_0$ in positive time if it is a sink (or in negative time if it is a source). Assume, without loss in generality, that $z_0$ is a sink. Then $P \cup \{z_0\}$ is a simply connected open subset of $\mathbb{C}$ and $\partial P$ consists of an at most countable union of closed connected subsets each being of one of three types: (1) zeros $z_1$ each with an attached orbit $\gamma_1$ such that $L_\alpha(\gamma_1) = z_1$ and $L_\omega(\gamma_1) = \infty$, (2) zeros $z_2$ each with an attached pair of distinct orbits $u, v$ with $L_\alpha(u) = L_\alpha(v) = z_2$ and $L_\omega(u) = L_\omega(v) = \infty$, and (3) orbits of the form $\gamma_\lambda$ where each $\gamma_\lambda$ is a positive and negative separatrix.

Proof. 1. $P$ is open: this follows from the continuous dependence of the flow on initial conditions.

2. $P$ is connected: because $P \cup \{z_0\}$ is path connected and contains a neighbourhood $B_\epsilon(z_0)$, $P$ is (path) connected.

3. If $B = \partial P$ then $B$ is closed. The point $z_0 \in B$.

4. $B$ consists of the disjoint union of closed connected subsets, each being the union of zeros and orbits: if

$$B = \bigcup_{\lambda \in \Lambda} B_\lambda$$

with each $B_\lambda$ connected, then $\overline{B_\lambda} \subset B$ is also connected. If $z \in \overline{B_\lambda} \setminus B_\lambda$ then, by the Poincaré-Bendixson theorem and the absence of limit cycles [3, Theorem 3.2], $f(z) = 0$ and $z \in B_\lambda$. Then each $B_\lambda$ can be expressed as the union of a chain of distinct orbits and zeros which is either finite or takes the form:

$$B_\lambda = \bigcup_{i \in \mathbb{Z}} \{z_i\} \cup \gamma_i$$

with $L_\alpha(g_i) = z_i, L_\omega(g_i) = z_{i+1}$ or $L_\alpha(g_i) = z_{i+1}, L_\omega(g_i) = z_i$ for each $i \in \mathbb{Z}$.

5. Each component of $B$ cannot contain more than one zero of $f(z)$: if the component $B_\lambda$ contains two (or more) distinct zeros then it must contain a section with three orbits and two zeros, say $(\alpha, z_1, \beta, z_2, \gamma)$. Neither $z_1$ nor $z_2$ can be a sink since they are on the boundary of a sink. If $z_1$ is a node (source) then the flow is hyperbolic in one sector at $z_2$ which is impossible for entire flows. The same applies to all of the other possible configurations.

6. If follows from 5. that each $B_\lambda$ must be one of the types (1), (2) or (3) given in the statement of the theorem. In case (1) and (2) since $L_\alpha(g) = z_1 \in \mathbb{C}$, and there is only one zero, we must have $L_\omega(g) = \infty$, so the orbits are unbounded. In case (3) the orbits must be unbounded in both time directions.
To show that each $B_\lambda$ is a separatrix consider type (1). The proofs for the other types are similar. Let $z_1$ be the associated zero and let $r > 0$ be sufficiently small that there are no other zeros in $B_r(z_o)$ or $B_r(z_1)$ and that these sets are disjoint. If Let $z_2$ be a point on $B_\lambda$ not in the closure of $B_r(z_1)$ and let $\epsilon > 0$ be given. There is an orbit $\gamma$ with $L_\alpha(\gamma) = z_1$, $L_\omega(\gamma) = z_o$ and $\gamma \cap B_\epsilon(z_2) \neq \emptyset$, which cuts the boundaries of the circular neighborhoods of $z_o$ and $z_1$. The rest of the proof is similar to that given for Theorem 3.1 (2) - it consists in showing that the time on any orbit starting at a point on the boundary of $B_r(z_o)$ and lying outside $\gamma$ (and hence tending to $B_\lambda$ is bounded by a fixed bound, not dependent on $z_2$.

7. $P \cup \{z_o\}$ is simply connected: if not there is a bounded re-gion with boundary meeting $B$ in a closed connected subset. This consists of an set of type (1),(2) or (3) so must be unbounded, a contradiction.

8. $|\Lambda| \leq n_o$: by embedding $\mathbb{C}$ in the Riemann sphere, the number of boundary components of types (2) and (3) are seen to be at most countable. Each component of type (1) is associated with a zero of $f(z)$, and there are an at most countable number of these, so they form an at most countable set also.

Figure 4 is a neighbourhood of a zero at 0 with 6 zeros on the boundary, all of type (1):

$$f(z) = (1 + i)z(z^2 + 1)(z^2 - 1)(z^2 - (i + 1)^2).$$

The separatrixes which tend to 0 (there are 6 of these also), are all included in the interior of the neighbourhood.

Remark 3.4. (1) The reader might suspect that for each of the results given in this paper, $|\Lambda| < \infty$. It would be well to keep in mind entire functions like:

$$f(z) = \alpha z^n \prod_{j \in \mathbb{N}} \left( \prod_{q \in F_j} \left(1 - \frac{z}{\frac{1}{j}e^{2\pi i q}}\right) \right) \exp(g_j(z))$$

where, for each $j$, the $g_j(z)$ are functions chosen to make the product converge, where the $F_j$ are the positive Farey rationals other than zero, where the integer $n$ has $n \geq 1$, and where $\alpha$ is a complex number of unit modulus. For example when $\alpha = i, n = 1$, the flow has a center at 0 with $|\Lambda| = n_o$. 

□
(2) When a separatrix occurs on the boundary of a sector at a zero then, in the main, that separatrix is associated also with another zero or zeros. Attempts to prove this in general, using devices such as the time advance map for the flow, the Riemann mapping theorem, Schwarz reflection, and properties of conformal maps [11, 12], have not succeeded. The example $\dot{z} = z \exp(z)$ (Figure 2) shows that a boundary separatrix need not be associated with any other finite zero.

(3) It is expected that a deeper understanding of separatrices might be found by considering the singularities of the flow with complex time $s$:

$$\frac{\partial \gamma(z,s)}{\partial s} = f(\gamma(z,s)), \gamma(z,0) = z$$

where $\gamma : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a locally holomorphic function of two complex variables, and where $f(z)$ is an entire function of one complex variable. An example of a modern reference to this very classical topic would be the work of the Costin’s including [5].

(4) It is conjectured that for entire flows, the vector field is complete and time advance map holomorphic on a dense open subset
of \( \mathbb{C} \). This would follow if it could be shown that, for each \( n \in \mathbb{N} \), the sets:

\[
F^n = \{ z : (\alpha_z, \beta_z) \subset (-\infty, n] \}, \quad F_n = \{ z : (\alpha_z, \beta_z) \subset [-n, \infty) \}
\]

have empty interior.

**References**


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