Old and new arithmetic and analytic equivalences of the Riemann hypothesis

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Progress on the problem during the twentieth century

• Harald Bohr and Edmund Landau in 1914 showed that the greater proportion of zeros were on or near the line.
• Hardy in 1914 proved that there were an infinite number of zeros on the line.
• Arne Beurling in 1942 developed the Hardy-Littlewood method further to show that a very small positive proportion of the zeros were on the line.
• Norman Levinson, dying of cancer in the early 1970’s, proved that more than \( \frac{1}{2} \) of the zeros were on the line.
• Brian Conrey, showed in 1987 that more than \( \frac{1}{2} \) of the zeros were on the line.
• By 2004 it had been shown by Xavier Gourdon that the first 10\(^{13}\) non-trivial zeros were on the line.

RH equivalences - why are they useful?

• They show the ubiquitous nature of RH
• The provide potential paths to resolving RH sometimes in completely different fields
• If RH is proved true, all of the equivalent statements are also true
• If RH is proved false, the negation of each of the equivalent statements is true

A study of the main equivalences of RH through 2017

Examples of arithmetic equivalences

Dixon and Spira’s inequality of 1965:

If \( s = \sigma + it \) with \( \sigma > 0.5 \) and \( t \geq 6.29073 \) then \( |\zeta(1-s)| > |\zeta(s)| \iff \text{RH} \).

Nicolas’ inequality of 1983 and 2012:

if \( \beta = 2 + \gamma - \log \pi - 2 \log 2 \)

\[
\kappa \Phi(s) < e^\gamma \log s \cdot \zeta(1+\beta), \quad s > 2 \iff \text{RH}.
\]

Robin’s inequality of 1984:

\[
\sigma(n) < e^\gamma \log n, \quad n > 5040 \iff \text{RH}.
\]

Some other arithmetic equivalences:

• The deviation of Farey fractions from a uniform distribution (Frate/Hlandau)
• The average order of the sum of the Mobius \( \mu(n) \) function (Littlewood)
• The determinant of the Redheffer divisibility matrix (Redheffer)
• The maximum order of the element of the symmetric group (Massias/Nicolas/Robinet)

Examples of analytic equivalences

The Riesz series criterion of 1915:

\( \forall \epsilon > 0, \quad \text{as} \quad s \to \infty, \quad \sum_{n=1}^{\infty} \left( \frac{1}{n^{1+\epsilon}} \right) \zeta(1+2\pi i n) e^{2\pi i n x} = o(1) \iff \text{RH} \).

Nyman-Beurling equivalence of 1955:

Let \( \lambda > 0 \) and let \( \xi(s) = \lambda \) be the fractional part of \( s \), so \( s = \lambda + \xi(s) \). Define a real linear space of real functions

\[
\mathcal{M} = \{ f : f(s) = \sum_{\epsilon \in \mathbb{C}} a_{\epsilon} \zeta(1+2\pi i \epsilon) e^{2\pi i \epsilon s}, \quad a_{\epsilon} \in \mathbb{R}, \quad \lambda \in (0,1], \quad \sum a_{\epsilon} = 0, \quad N \in \mathbb{N} \}.
\]

Let \( 1 \leq p < \infty \). The subspace \( \mathcal{M} \) is dense in the Banach space \( L^p(0,1) \) if and only if \( \xi(s) \) has no zeros in the right half plane \( \sigma > 1 \).

If \( 0 < \sigma < 1 \) and \( f \in \mathcal{M} \), we have

\[
\xi(s) = -\frac{1}{s-1} \int_0^\lambda f(x) x^{-s} dx = \frac{1}{s-1} \sum a_{\epsilon} \Lambda(\epsilon).
\]

Some other analytic equivalences:

• \( \left| \frac{\log(1+it)}{t} \right| \) is monotonic for \( \sigma > 0.5 \) and all fixed \( t \) (Sound/Dumitrescu).
• For all \( n \in \mathbb{Z} \) we have \( \sum_{n=1}^{\infty} \left( 1 - \frac{1}{2} \right) > 0 \) (Li, Bombieri/Lagarias).
• Integral equations (Sekatskii/Beltraminelli/Merlini, Salem, Levinson).
• Equivalences based on Weil’s explicit formula (Weil).
• Discrete measure equivalences (Venkovsky).
• Smooth number equivalence (Hildebrand).
• Hermitian form equivalence (Yoshida), with insufficient room for others ....

Overview of discoverers of (most of) the main equivalences

New and evolving equivalences

The new Tao/Rogers equivalence: \( \Lambda = 0 \)

The Newman-de Bruijn constant \( \Lambda \) is the minimum value of \( \lambda \) such that if

\[ Z(s) - \frac{\xi(1-s)}{\zeta(1-s)} = \int_{1-\epsilon}^1 \Phi(t) e^{\pi i t} dt, \quad \frac{\xi(1-s)}{\zeta(1-s)} = \int_{1-\epsilon}^1 e^{\pi i t} dt, \quad \Phi(t) = e^{\pi i t}.
\]

has only real zeros. C. M. Newman in 1976 showed \( \Lambda < \Lambda \) and conjectured \( \Lambda = 0 \). Then RH is equivalent to \( \Lambda \leq 0 \). Brad Rogers and Terence Tao have shown recently that Newman’s conjecture is true, and so RH is equivalent to \( \Lambda = 0 \), whatever definition of \( \Lambda \) is used. A preprint is “The de Bruijn-Newman constant is non-negative”, arXiv: 1801.05914v2 [math.NT] (14 Feb 2018).

The recently announced result of Ono/Griffen/Rolen/Zagier

Jensen polynomials are of the form

\[ f^{(n)}(x) = \sum_{i=0}^n \binom{n}{i} y_i x^i \]

where the \( y_i \) are associated with the Taylor expansion of \( Z(s) \), i.e. they satisfy

\[ Z(s) = \frac{1}{s-1} \sum_{n=1}^{\infty} \frac{\xi(1-s)}{\zeta(1-s)} e^{2\pi i n x}.
\]

Ken Ono, Michael Griffin, Larry Rolen and Don Zagier have shown that all but a finite number of the Jensen polynomials for the Riemann Xi function are hyperbolic, i.e. have all real zeros. A 1927 result of Polya is apparently that RH is equivalent to all of these polynomials being hyperbolic. This hyperbolicity has been proved for degrees \( d \leq 3 \). They obtained an arbitrary precision asymptotic formula for the derivatives \( Z^{(d)}(0) \), which allows them to prove the hyperbolicity of 100% of the Jensen polynomials of each degree. They used Hermite polynomials.

Is RH undecidable?

If RH is undecidable, then it is and is false there is a zero (which we cannot find) off the critical line. This zero would provide a proof that RH is false, but there is no such proof since RH is undecidable. Therefore it is true but can never be proved!

Is it computable? If RH is false then a finite computation will find a integer which contradicts Robin’s (or easier Shapiros) inequality, so at least its semi-computable.

A likely equivalence of Bombieri

If \( \theta(t) \) is the (continuous) argument of \( Z(s) \) on the critical line then \( Z(s) = e^{\theta(t)} \zeta(1+it) \) is real and has the same zeros as \( \xi(s) \) on the line. If \( Z(1/2) \) has a positive local minimum or negative local maximum then RH would be false. It is conjectured that this statement has a converse, which if true would provide a valuable RH equivalence.