Lecture 10: Taylor Expansions

Theorem 30: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f:[a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b) i.e. $f^{(n)}$ exists on (a,b). Then $\exists\ \xi \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(\xi)$$

X

$$= \sum_{j=0}^{n-1} \frac{(b-a)^j}{j!} f^{(j)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(\xi)$$

Proof Define a real number K by the identity

$$\frac{(b-a)^n}{n!}K = f(b) - f(a) - \sum_{j=1}^{n-1}\frac{(b-a)^j}{j!}\cancel{f(a)^{(j)}}$$

$\boxed{f^{(j)}(a)}$ X

and define a function φ on $[a,b]$ by

$$\varphi(x) = f(b) - f(x) - \sum_{j=1}^{n-1}\frac{(b-x)^j}{j!}f^{(j)}(x) - \frac{(b-x)^n}{n!}K$$

Then $\varphi(a) = 0$ by the definition of the number K and $\varphi(b) = f(b) - f(b) - 0 - 0 = 0$

φ is continuous on $[a,b]$ since the $f^{(j)}$ are continuous for $0 \leq j \leq n-1$ and φ is differentiable on (a,b) since $f^{(n)}$ exists on (a,b). Therefore we may apply Rolle's Theorem to φ to infer the existence of a number ξ in (a,b) with $\varphi'(\xi) = 0$.

$$\text{Now } \varphi'(x) = 0 - f'(x) - \sum_{j=1}^{n-1}\left[\frac{-j(b-x)^{j-1}}{j!}f^{(j)}(x) + \frac{(b-x)^j}{j!}f^{(j+1)}(x)\right] - \frac{-n(b-x)^{n-1}}{n!}K$$

$$\Rightarrow \varphi'(x) = -f'(x) - \Big[-f'(x) + \frac{(b-x)}{1}f^{(2)}(x) - \frac{(b-x)^{1}}{1}f^{(2)}(x) + \frac{(b-x)^2}{2!}f^{(3)}(x)$$

X

$$\cdots - \frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x) + \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)\Big] + \frac{(b-x)^{n-1}}{(n-1)!}K$$

Therefore

$$0 = \varphi'(\xi) = -\frac{(b-\xi)^{n-1}}{(n-1)!}f^{(n)}(\xi) + \frac{(b-\xi)^{n-1}}{(n-1)!}K$$

and so $K = f^{(n)}(\xi)$. Putting this expression back into the defining expression for K gives the Taylor expansion and completes the proof.

If $b < a$ then the expansion still exists with $\xi \in (b,a)$.

If $b = a + h$ then the expansion may be written

$$f(a+h) = f(a) + \sum_{j=1}^{n-1}\frac{f^{(a)}(j)}{j!}h^j + \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

$\boxed{f^{(j)}(a)}$ X

for some θ (depending on a, h, f) with $|\theta| < 1$.

The reader will note that Rolle's Theorem is used in the proof of Taylor's Theorem. Some regard the latter as the most important theorem in analysis because of its use in obtaining approximations and deriving power series (see later). However it appears that