Chapter 17

Uniformly Continuous Functions

In this chapter, we shall discuss uniformly continuous functions having metric spaces for domain and range. We shall also prove that any metric space can be embedded in a complete metric space.

We denote by (X_1, d_1) and (X_2, d_2) two metric spaces.

17.1 Definition.—*We say that $f: X_1 \to X_2$ is uniformly continuous on X_1 if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$d_2(f(x_1), f(x_2)) < \varepsilon$$

whenever x_1 and x_2 belong to X_1 and $d_1(x_1, x_2) < \delta$.

We obtain, obviously, an *equivalent* definition if "$d_2(f(x_1), f(x_2)) < \varepsilon$ and $d_1(x_1, x_2) < \delta$" are replaced by

$$\text{“}d_2(f(x_1), f(x_2)) \leq \varepsilon \quad \text{and} \quad d_1(x_1, x_2) \leq \delta.\text{”}$$

Since δ depends on ε, we often use some notation to indicate this. For instance, we may write δ_ε or $\delta(\varepsilon)$ instead of δ.

If $A \subset X_1$, we say that $f: A \to X_2$ is *uniformly continuous on A* if considered as a mapping "of the metric space* $(A, (d_1)_A)$ into (X_2, d_2)," f is uniformly continuous in the sense of Definition 17.1.

Hence, the function $f: A \to X_2$ is uniformly continuous on A if and only if, for every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$d_2(f(x_1), f(x_2)) < \varepsilon$$

whenever x_1 and x_2 belong to A and $d(x_1, x_2) < \delta$.

* Recall that $(d_1)_A = d \mid A \times A$.

17.2 Theorem.—*Let $A \subset X_1$ and let $f: A \to X_2$ be uniformly continuous. Then f is continuous on A.*

Proof.—Let $\varepsilon > 0$. Since f is uniformly continuous, there is $\delta_\varepsilon > 0$ such that

$$d_2(f(x_1), f(x_2)) < \varepsilon$$

whenever x_1 and x_2 belong to A and $d(x_1, x_2) < \delta_\varepsilon$. Let $a \in A$; we deduce that

$$d_2(f(a), f(x)) < \varepsilon$$

whenever $x \in A$ and $d(a, x) < \delta_\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we deduce that f is continuous at a (see 13.18). Since $a \in A$ was arbitrary, f is continuous on A.

We have shown, therefore, that a uniformly continuous function is continuous. We shall see in Example 2 that the converse is not true; that is, that there exist continuous functions that are not uniformly continuous.

If we compare 17.1 with the definition of continuity, we see that in the definition of continuity of f at a (see, for instance, 13.18), given $\varepsilon > 0$, the corresponding neighborhood of a depends not only on ε but also on $a \in A$ (although we usually do not indicate this dependence by any supplementary symbol). Expressed with inequalities, this means that "δ_ε" depends not only on ε, but also on a. In the definition of uniform continuity, δ_ε *does not depend on a.*

Example 1.—Let (X_1, d_1) and (X_2, d_2) be two metric spaces, $A \subset X_1$ totally bounded and $f: A \to X_2$ uniformly continuous. Then the set $f(A)$ is *bounded.*

Since f is uniformly continuous, there is $\delta > 0$ such that

$$d_2(f(x), f(y)) < 1$$

whenever x and y belong to A and $d_1(x, y) < \delta$. Since A is totally bounded, there is a finite family $(a_j)_{1 \leq j \leq n}$ of elements of A such that $(V_\delta(a_j))_{1 \leq j \leq n}$ is a covering of A. Let

$$L = \sup \{d(f(a_i), f(a_j)) \mid 1 \leq i \leq n,\ 1 \leq j \leq n\}.$$

If x and y belong to A, there are $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $x \in V_\delta(a_i)$ and $y \in V_\delta(a_j)$. Hence

$$d(f(x), f(y)) \leq d(f(x), f(a_i)) + d(f(a_i), f(a_j)) + d(f(a_j), f(y)) \leq 2 + L.$$

Since x and y were arbitrary elements of A, we deduce that $\delta(f(A)) \leq 2 + L$. Hence, $f(A)$ is bounded.

If (X_2, d_2) is the metric space $\mathbf{R}$, then $f(A)$ is bounded if and only if there is $K \in \mathbf{R}_+$ such that $|f(x)| \leq K$ for all $x \in A$.

Example 2.—The mapping $x \mapsto 1/x$ of $(0, 1)$ into $\mathbf{R}$ is *not* uniformly continuous.

Let $L \geq 0$. By Archimedes' property, there is $p \in \mathbf{N}$ satisfying $p > L$. Hence $x_0 \in (0, 1)$ and $|1/x_0| = p > L$ if $x_0 = 1/p$. Hence there is no L satisfying $|1/x| \leq L$ for all $x \in (0, 1)$, and hence (by the result in Example 1), the mapping $x \mapsto 1/x$ of $(0, 1)$ into $\mathbf{R}$ is not uniformly continuous.

Example 3.—The mapping $x \mapsto 1/x$ of $\mathbf{R}^*$ into $\mathbf{R}$ is not uniformly continuous.

We leave the details to the reader.

17.3 Theorem.—*Let (X_1, d_1) and (X_2, d_2) be two metric spaces, $A \subset X_1$ compact and $f: A \to X_2$ continuous. Then f is uniformly continuous.*

Proof.—Let $\varepsilon > 0$ and let $t \in A$. Since $f: A \to X_2$ is continuous, there is $\delta(t) > 0$ such that $x \in V_{\delta(t)}(t)$ implies that

$$d_2(f(x), f(t)) < \varepsilon/2.$$

Clearly, $(V_{\delta(t)/2}(t))_{t \in A}$ is an *open* covering of A. Since A is compact, there exists a finite family $(t_j)_{1 \leq j \leq n}$ of elements of A such that $(V_{\delta(t_j)/2}(t_j))_{1 \leq j \leq n}$ is a covering of A. Let $\delta = \inf\{\delta(t_1)/2, \ldots, \delta(t_n)/2\}$. Now let x and y in A be such that $d(x,y) < \delta$. Since $(V_{\delta(t_j)/2}(t_j))_{1 \leq j \leq n}$ is a covering of A, there exists $1 \leq j_0 \leq n$ such that $x \in V_{\delta(t_{j_0})/2}(t_{j_0})$. Hence,

$$d_1(t_{j_0}, x) < \delta(t_{j_0})/2 < \delta(t_{j_0})$$

and

$$\begin{aligned} d_1(t_{j_0}, y) &< d_1(t_{j_0}, x) + d_1(x, y) < \delta(t_{j_0})/2 + \delta \\ &< \delta(t_{j_0})/2 + \delta(t_{j_0})/2 = \delta(t_{j_0}). \end{aligned}$$

Hence, $d_1(t_{j_0}, x) < \delta(t_{j_0})$ and $d_1(t_{j_0}, y) < \delta(t_{j_0})$. We obtain, then,

$$\begin{aligned} d_2(f(x), f(y)) &< d_2(f(x), f(t_{j_0})) + d_2(f(t_{j_0}), f(y)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that f is uniformly continuous.

Exercise.—Let X be a metric space, $A \subset X$, and let $C^u_{\mathbf{R}}(A)$ be the set of all mappings $f: A \to \mathbf{R}$ uniformly continuous on A. (a) Show that, endowed with the usual addition and scalar multiplication, $C^u_{\mathbf{R}}(A)$ is a vector space. (b) Show that if f and g belong to $C^u_{\mathbf{R}}(A)$, it is not necessarily true that fg belongs to A. (Hint: Let $A = X = \mathbf{R}$ and let j be the mapping $x \mapsto x$ of $\mathbf{R}$ into $\mathbf{R}$; then $j \in C^u_{\mathbf{R}}(A)$, whereas $j^2 \notin C^u_{\mathbf{R}}(A)$.)

Again, let (X_1, d_1) and (X_2, d_2) be two metric spaces.

17.4 Definition.—*Let $A \subset X_1$ and $f: A \to X_2$. We say that f is a Lipschitz function* if there is $L \geq 0$ such that*

$$d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all x and y in A.

It is easy to see that a Lipschitz function is uniformly continuous. In fact, let $\varepsilon > 0$ and let $\delta_\varepsilon = \varepsilon/(L + 1)$. Then x and y in A and $d_1(x, y) \leq \delta_\varepsilon$ imply that

$$d_2(f(x), f(y)) \leq L\varepsilon/(L + 1) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that f is uniformly continuous on A.

Example 4.—For each non-void set $A \subset X_1$, the mapping $f: x \mapsto d_1(x, A)$ of X_1 into $\mathbf{R}$ is a Lipschitz function.

In fact, (see 13.6),

$$|f(x) - f(y)| \leq d_1(x, y)$$

for all x and y in X_1.

Recall that we have already proved that f is continuous on X_1 (see 13.19).

Before proceeding farther, we shall make the following remarks: Let (X, d) be a metric space, $(x_n)_{n \in \mathbf{N}}$ a sequence of elements of X that converges to a, and $(y_n)_{n \in \mathbf{N}}$ a sequence of elements of X that converges to b. Then:

17.5(i) $\lim_{n \in \mathbf{N}} d(x_n, y_n) = d(a, b)$;

(ii) $a = b \Leftrightarrow \lim_{n \in \mathbf{N}} d(x_n, y_n) = 0.$

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* Lipschitz functions having for domain a part of $\mathbf{R}$ have been introduced in Example 5, Chapter 5.

We leave to the reader the proof of 17.5 (hint: Use Theorem 13.21).

Let (X_1, d_1) and (X_2, d_2) be two metric spaces.

17.6 Definition.—Suppose $A \subset X_1$. *A mapping* $\varphi: A \to X_2$ *is an isometry if*

$$d_2(\varphi(x), \varphi(y)) = d_1(x, y)$$

for all x *and* y *belonging to* A.

The metric spaces (X_1, d_1) and (X_2, d_2) are said to be *isometric* if there is an isometry of X_1 onto X_2 or an isometry of X_2 onto X_1.
We notice that:

17.7 *If* $\varphi: A \to X_2$ *is an isometry, then* φ *is an injection.*

In fact, if x and y belong to A, and $x \neq y$, then

$$d_2(\varphi(x), \varphi(y)) = d_1(x, y) \neq 0,$$

so that $\varphi(x) \neq \varphi(y)$. Since x and y were arbitrary, φ is injective.

17.8 *If* $\varphi: A \to X_2$ *is an isometry, then* φ *is a Lipschitz function.*

17.9 Let $\varphi: A \to X_2$ be an isometry. Consider φ as a bijection of A onto $\varphi(A)$. If φ^{-1} is the inverse of this mapping, then $\psi: x \mapsto \varphi^{-1}(x)$, *on* $\varphi(A)$ *to* X_1, *is an isometry.*

17.10 Let $\varphi: A \to X_2$ be an isometry and $(x_n)_{n \in \mathbf{N}}$ a sequence of elements of A. Then $(x_n)_{n \in \mathbf{N}}$ *is a Cauchy sequence if and only if* $(\varphi(x_n))_{n \in \mathbf{N}}$ *is a Cauchy sequence.*

This follows immediately from the equation,

$$d_2(\varphi(x_n), \varphi(x_m)) = d_1(x_n, x_m)$$

for $n \in \mathbf{N}$, $m \in \mathbf{N}$.

17.11 Let $\varphi: A \to X_2$ be an isometry, $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ two sequences of elements of A that converge to an element* $a \in X_1$. Assume that (X_2, d_2) is *complete*. Then $(\varphi(x_n))_{n \in \mathbf{N}}$ and $(\varphi(y_n))_{n \in \mathbf{N}}$ *are convergent and*

(*) $\lim_{n \in \mathbf{N}} \varphi(x_n) = \lim_{n \in \mathbf{N}} \varphi(y_n)$.

* Notice that we do not assume that $a \in A$.

Since $(x_n)_{n \in \mathbf{N}}$ is convergent, it is Cauchy. By 17.10, $(\varphi(x_n))_{n \in \mathbf{N}}$ is a Cauchy sequence. Since (X_2, d_2) is complete, $(\varphi(x_n))_{n \in \mathbf{N}}$ is convergent. In the same way, we show that $(\varphi(y_n))_{n \in \mathbf{N}}$ is convergent. Since

$$\lim_{n \in \mathbf{N}} x_n = \lim_{n \in \mathbf{N}} y_n = a,$$

we deduce from 17.5(ii) that

$$\lim_{n \in \mathbf{N}} d_1(x_n, y_n) = 0.$$

Since φ is an isometry, we have

$$\lim_{n \in \mathbf{N}} d_2(\varphi(x_n), \varphi(y_n)) = 0.$$

Using 17.5(ii), again, we conclude that

$$\lim_{n \in \mathbf{N}} \varphi(x_n) = \lim_{n \in \mathbf{N}} \varphi(y_n).$$

17.12 Theorem.—*Let* (X_1, d_1) *and* (X_2, d_2) *be two complete metric spaces,* A_1 *a dense part of* X_1, A_2 *a dense part of* X_2, *and* $\varphi: A_1 \to X_2$ *an isometry such that* $\varphi(A_1) = A_2$. *Then there exists an isometry* $\bar{\varphi}$ *of* X_1 *onto* X_2 *such that* $\bar{\varphi} \mid A_1 = \varphi$.

Proof.—Let $x \in X_1$ and let $(x_n)_{n \in \mathbf{N}}$ be a sequence of elements of A_1 that converges to x. Then $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence. By 17.10, $(\varphi(x_n))_{n \in \mathbf{N}}$ is a Cauchy sequence; since (X_2, d_2) is complete, $(\varphi(x_n))_{n \in \mathbf{N}}$ is convergent. Let

$$\bar{\varphi}(x) = \lim_{n \in \mathbf{N}} \varphi(x_n).$$

By 17.11, $\bar{\varphi}(x)$ does not depend on the particular sequence that converges to x. Since $x \in X_1$ was arbitrary, we defined this way a mapping $\bar{\varphi}: X_1 \to X_2$. Clearly, $\bar{\varphi} \mid A_1 = \varphi$.
Let x and y be two elements belonging to X_1. Let $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ be sequences of elements of A_1 that converge to x and y, respectively. Using 17.5(i), we obtain

$$\begin{aligned} d_2(\bar{\varphi}(x), \bar{\varphi}(y)) &= \lim_{n \in \mathbf{N}} d_2(\bar{\varphi}(x_n), \bar{\varphi}(y_n)) \\ &= \lim_{n \in \mathbf{N}} d_1(x_n, y_n) = d_1(x, y). \end{aligned}$$

Since x and y were arbitrary, it follows that $\bar{\varphi}$ is an isometry.
Now let $y \in X_2$ and let $(y_n)_{n \in \mathbf{N}}$ be a sequence of elements of A_2 that converges to y. Since $\varphi(A_1) = A_2$, we deduce that for each $n \in \mathbf{N}$ there is $x_n \in A_1$ such that $\varphi(x_n) = y_n$. By 17.10, $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence. Since (x_1, d_1) is complete, $(x_n)_{n \in \mathbf{N}}$ converges to some

element $x \in X_1$. We deduce that

$$y = \lim_{n \in \mathbf{N}} \varphi(x_n) = \bar{\varphi}(x).$$

Since $y \in X_2$ was arbitrary, we conclude that $\bar{\varphi}(x_1) = x_2$.

The uniqueness of $\bar{\varphi}$ follows from the fact that $\bar{A}_1 = X_1$ and from Corollary 7.21.

Hence, Theorem 17.2 is completely proved.

Exercise.—Let (X_1, d_1) and (X_2, d_2) be two metric spaces, A_1 a dense part of X_1 and $f: A_1 \to X_2$ a uniformly continuous function. Suppose that (X_2, d_2) is *complete*. Then there exists a unique uniformly continuous mapping $\bar{f}: X_1 \to X_2$ such that $\bar{f} \mid A_1 = f$.

We shall prove the result stated at the end of Chapter 15; namely, that any metric space can be embedded in a complete metric space. We shall first establish certain results that will be used in the proof.

Let (X, d) be a metric space and let $\mathscr{C}$ be the set of all *Cauchy sequences* of elements of X. For

$$\mathbf{x} = (x_n)_{n \in \mathbf{N}} \in \mathscr{C} \quad \text{and} \quad \mathbf{y} = (y_n)_{n \in \mathbf{N}} \in \mathscr{C},$$

we write

$$\mathbf{x} \equiv \mathbf{y}(S) \Leftrightarrow \lim_{n \in \mathbf{N}} d(x_n, y_n) = 0;$$

we define thus an *equivalence relation* in $\mathscr{C}$.

We shall now show that:

17.13 *If* $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$ *and* $\mathbf{y} = (y_n)_{n \in \mathbf{N}}$ *are elements of* $\mathscr{C}$, *then* $(d(x_n, y_n))_{n \in \mathbf{N}}$ *is convergent.*

17.14 *If* $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$, $\mathbf{x}' = (x'_n)_{n \in \mathbf{N}}$, $\mathbf{y} = (y_n)_{n \in \mathbf{N}}$, *and* $\mathbf{y}' = (y'_n)_{n \in \mathbf{N}}$ *are elements of* $\mathscr{C}$, *and if* $\mathbf{x} \equiv \mathbf{x}'$ *and* $\mathbf{y} \equiv \mathbf{y}'$, *then*

$$\lim_{n \in \mathbf{N}} d(x_n, y_n) = \lim_{n \in \mathbf{N}} d(x'_n, y'_n).$$

Proof of 17.13.—For all $n \in \mathbf{N}$ and $m \in \mathbf{N}$, we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m);$$

hence

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m).$$

We deduce that $(d(x_n, y_n))_{n \in \mathbf{N}}$ is a Cauchy sequence, and hence convergent.

Proof of 17.14.—For all $n \in \mathbf{N}$ and $m \in \mathbf{N}$, we have

$$d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n)$$

and

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n);$$

hence

$$|d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n).$$

We deduce that

$$\lim_{n \in \mathbf{N}} |d(x_n, y_n) - d(x'_n, y'_n)| = 0.$$

By 17.13, $(d(x_n, y_n))_{n \in \mathbf{N}}$ and $(d(x'_n, y'_n))_{n \in \mathbf{N}}$ are convergent, and then, by 17.5(ii),

$$\lim_{n \in \mathbf{N}} d(x_n, y_n) = \lim_{n \in \mathbf{N}} d(x'_n, y'_n).$$

17.15 Theorem.—*Let* (X, d) *be a metric space. Then there exists a complete metric space* $(\hat{X}, \hat{d})$ *and an isometry* φ *of* X *into* $\hat{X}$ *such that* $\overline{\varphi(X)} = \hat{X}$.

Proof.—Let $\hat{X} = \mathscr{C}/S$ and let $\mathbf{x} \mapsto \hat{\mathbf{x}}$ be the canonical mapping of $\mathscr{C}$ onto $\hat{X}$.

If $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ belong to $\hat{X}$ (so that $\mathbf{x} = (x_n)_{n \in \mathbf{N}} \in \mathscr{C}$ and $\mathbf{y} = (y_n)_{n \in \mathbf{N}} \in \mathscr{C}$), we define

$$\hat{d}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \lim_{n \in \mathbf{N}} d(x_n, y_n).$$

By 17.13 and 17.14, $\hat{d}$ is well-defined as a mapping of $\hat{X} \times \hat{X}$ into $\mathbf{R}$.

It is easy to see that $\hat{d}$ is a *metric* on $\hat{X}$. *Thus the metric space* $(\hat{X}, \hat{d})$ *is defined.*

For each $x \in X$, let $\mathbf{x}^*$ be the Cauchy sequence $(x_n)_{n \in \mathbf{N}}$ where $x_n = x$ for all $n \in \mathbf{N}$. Let $\varphi(x) = \hat{\mathbf{x}}^*$ for $x \in X$. If x and y belong to X, we deduce, from the definition of $\hat{d}$, that

$$\hat{d}(\varphi(x), \varphi(y)) = \hat{d}(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*) = d(x, y).$$

Since x and y were arbitrary, $\varphi: X \to \hat{X}$ is an isometry.

Now let $\hat{\mathbf{x}} \in \hat{X}$, where $\mathbf{x} = (x_n)_{n \in \mathbf{N}}$, and let $\varepsilon > 0$. Since $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, there is $p \in \mathbf{N}$ such that $m \geq p$ and $n \geq p$ imply that

$$d(x_m, x_n) \leq \varepsilon.$$

Then

$$\hat{d}(\hat{\mathbf{x}}, \hat{\mathbf{x}}_p^*) = \lim_{n \in \mathbf{N}} d(x_n, x_p) \leq \varepsilon.$$

Since $\hat{\mathbf{x}} \in \hat{X}$ and $\varepsilon > 0$, we deduce that $\overline{\varphi(X)} = \hat{X}$.

Thus the isometry $\varphi: X \to \hat{X}$ is defined and we have shown that $\overline{\varphi(X)} = \hat{X}$.

Now let $(\hat{\mathbf{x}}_n)_{n\in\mathbf{N}}$ be a Cauchy sequence of elements of $\hat{X}$. For each $n \in \mathbf{N}$, let $y_n \in X$ be such that

$$\hat{d}(\hat{\mathbf{x}}_n, \varphi(y_n)) \leq 1/n.$$

Then $(\varphi(y_n))_{n\in\mathbf{N}}$ is a Cauchy sequence of elements of $\hat{X}$. Since φ is an isometry, we deduce (see 17.10) that $\mathbf{y} = (y_n)_{n\in\mathbf{N}}$ is a Cauchy sequence of elements of X.

Let $\varepsilon > 0$ and let $n_\varepsilon \in \mathbf{N}$ be such that $1/n_\varepsilon \leq \varepsilon/2$ and $d(y_n, y_m) \leq \varepsilon/2$ if $n \geq n_\varepsilon$, $m \geq n_\varepsilon$. Then, if $n \geq n_\varepsilon$

$$\begin{aligned}
\hat{d}(\hat{\mathbf{x}}_n, \hat{\mathbf{y}}) &\leq \hat{d}(\hat{\mathbf{x}}_n, \varphi(y_n)) + \hat{d}(\varphi(y_n), \hat{\mathbf{y}}) \\
&\leq \frac{1}{n} + \hat{d}(\varphi(y_n), \hat{\mathbf{y}}) \\
&= \frac{1}{n} + \lim_{m\in\mathbf{N}} d(y_n, y_m) \\
&\leq 2(\varepsilon/2) = \varepsilon.
\end{aligned}$$

Hence, $\hat{d}(\hat{\mathbf{x}}_n, \hat{\mathbf{y}}) \leq \varepsilon$ if $n \geq n_\varepsilon$; since $\varepsilon > 0$ was arbitrary, $(\hat{\mathbf{x}}_n)_{n\in\mathbf{N}}$ converges to $\hat{\mathbf{y}}$.

Thus we have shown that $(\hat{X}, \hat{d})$ is complete, and hence Theorem 17.15 is proved.

Remarks.—The metric space $(\hat{X}, \hat{d})$ is called the *completion* of (X, d). If we *identify* X with $\varphi(X)$, then:

(i) $X \subset \hat{X}$;

(ii) $\hat{d} \mid X \times X = d$;

(iii) $\overline{X} = \hat{X}$.

We shall show below that, in a certain sense, $(\hat{X}, \hat{d})$ is "unique."

17.16 Theorem.—*Let (X, d) be a metric space, (X', d') a complete metric space, φ' an isometry of X into X' such that $\overline{\varphi'(X)} = X'$, (X'', d'') a complete metric space, and φ'' an isometry of X into X'' such that $\overline{\varphi''(X)} = X''$. Then (X', d') and (X'', d'') are isometric.*

Proof.—Let $A' = \varphi'(X)$, $A'' = \varphi''(X)$, and let $\psi: A' \to A''$ be defined by

$$\psi(\varphi'(x)) = \varphi''(x)$$

for $x \in X$. Then ψ is an isometry of A' into X'' such that $\psi(A') = A''$. By Theorem 17.12, (X', d') and (X'', d'') are isometric.

Exercises for Chapter 17

1. Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $f: X_1 \to X_2$ be a Lipschitz function. If A is a totally bounded subset of X_1, then $f(A)$ is totally bounded in X_2.

2. Let d be the usual metric on $\mathbf{R}$ and δ the metric on $\mathbf{R}$ defined in Example 6, Chapter 15. Show that the mapping $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x$ for $x \in \mathbf{R}$ is uniformly continuous as a mapping of $(\mathbf{R}, d)$ into $(\mathbf{R}, d)$, but is *not* uniformly continuous as a mapping of $(\mathbf{R}, \delta)$ into $(\mathbf{R}, d)$.

3. Let f be the mapping $x \mapsto 1/x$ of $(0, \infty)$ into $\mathbf{R}$. Show that for every $n > 0$, $f \mid [\eta, +\infty)$ is uniformly continuous (although f is not uniformly continuous).

4. Generalize the results stated in Exercises 5, 6, and 9 at the end of Chapter 5 to Lipschitz functions having for domain a subset of a metric space and for range $\mathbf{R}$ (see Example 4).