Lecture 2 : Subsequences and Cauchy sequences in $\mathbb{R}$.

Let $(n_j)_{j\in\mathbb{N}}$ with $n_1 < n_2 < n_3 \dots$ $n_j \in \mathbb{N}$

and $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then $(a_{n_j})_{j\in\mathbb{N}}$

is called a <u>subsequence</u>.

<u>Recall</u> $\lim_{n\to\infty} a_n = \alpha$ means $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ so $n \geq N_\varepsilon \Rightarrow$
$|a_n - \alpha| < \varepsilon$. If so we write $a_n \to \alpha$.

<u>Theorem</u> $a_n \to \alpha$ implies any subsequence $a_{n_j} \to \alpha$.

<u>Proof</u> Given $\varepsilon > 0$, choose N_ε so $\forall n \geq N_\varepsilon$, $|a_n - \alpha| < \varepsilon$.
But (exercise) $\forall j$, $n_j \geq j$. Thus $\forall j \geq N_\varepsilon$ $n_j \geq j \geq N_\varepsilon$
$\Rightarrow |a_{n_j} - \alpha| < \varepsilon$ Hence $a_{n_j} \to \alpha$. //

<u>Note</u>: a non-convergent sequence may have convergent subsequences.
$(0, 1, 2, 0, 1, 2, 0, 1, 2, \dots)$
We say a sequence (a_n) is <u>increasing</u> if $\forall n \in \mathbb{N}$ $a_n \leq a_{n+1}$.
<u>Theorem</u> If a sequence (a_n) is increasing and bounded above
$(a_n \leq M\ \forall n \in \mathbb{N})$ then $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\} < \infty$.

<u>Proof</u> since $a_n \leq M$, M is an upper bound for the sequence values. Thus M is greater or equal to the least upper bound or sup. Hence $\alpha := \sup\{a_n : n\in\mathbb{N}\} < \infty$.
Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ so $\alpha - \varepsilon < a_N \leq \alpha$. If $n \geq N$

$$\alpha - \varepsilon < a_N \leq a_n \leq \alpha < \alpha + \varepsilon \Rightarrow |a_n - \alpha| < \varepsilon.$$

Therefore $\lim_{n\to\infty} a_n = \alpha$. //

Ex $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, …, $a_{n+1} = \sqrt{2+a_n}$
then (a_n) is increasing with limit 2.

Def^n We say a sequence (a_n) is Cauchy if $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$
so $|a_n - a_m| < \varepsilon$ $\forall n, m \geq N_\varepsilon$.

A Cauchy sequence has a small tail.

Theorem If $a_n \to \alpha$ then (a_n) is Cauchy.

Proof Given $\varepsilon > 0$ $\exists N_\varepsilon$ so $\forall n \geq N_\varepsilon$, $|a_n - \alpha| < \frac{\varepsilon}{2}$.
Thus $\forall n, m \geq N_\varepsilon$, $|a_n - a_m| \leq |a_n - \alpha| + |a_m - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
Therefore (a_n) is Cauchy. //

Any Cauchy sequence is bounded: Let $\varepsilon = 1$. $\forall n, m \geq N_1$
$|a_n - a_m| < 1 \Rightarrow |a_n| - |a_{N_1}| < 1$ $\forall n \geq N_1$, Hence

$$|a_n| \leq \underbrace{\max\{|a_1|, |a_2|, \ldots |a_{N_1}|, |a_{N_1}| + 1\}}_{=:M} \quad \forall n \in \mathbb{N}.$$

so (a_n) is bounded i.e.

$$-M \leq a_n \leq M \quad \forall n \in \mathbb{N}.$$

Theorem Let (a_n) be Cauchy. Let a subsequence $a_{n_j} \to \alpha$ in $\mathbb{R}$.
Then $a_n \to \alpha$ also.

Proof Given $\varepsilon > 0$ $\exists N_1$ so $\forall n, m \geq N_1$ $|a_n - a_m| < \frac{\varepsilon}{2}$. And
$\exists N_2$ so $|a_{n_j} - \alpha| < \frac{\varepsilon}{2}$ $\forall j \geq N_2$. Let $N_\varepsilon = \max\{N_1, N_2\}$
and choose j so $j \geq N_\varepsilon$.
If $n \geq N_\varepsilon$, $|a_n - \alpha| \leq |a_n - a_{n_j}| + |a_{n_j} - \alpha|$
But $n_j \geq j \geq N_\varepsilon \geq N_2$ and $n \geq N_\varepsilon \geq N_1$
$\Rightarrow$ $|a_n - \alpha| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $a_n \to \alpha$. //

Theorem Let (a_n) be Cauchy in $\mathbb{R}$. Then $\exists \alpha \in \mathbb{R}$ so $a_n \to \alpha$.

Proof (a_n) is bounded so $\exists m, M$ with $m \leq a_n \leq M \; \forall n$.

Let $b_n := \inf\{a_n, a_{n+1}, \ldots\}$, $c_n := \sup\{a_n, a_{n+1}, \ldots\}$

$\Rightarrow \quad m \leq b_n \leq c_n \leq M \quad \forall n \in \mathbb{N}$.

since $\{a_{n+1}, a_{n+2}, \ldots\} \subset \{a_n, a_{n+1}, \ldots\}$, $b_n \leq b_{n+1}$

$\Rightarrow$ (b_n) is increasing. It is also bounded above by M.

$\Rightarrow$ $\lim_{n\to\infty} b_n = \beta$ exists.

Also (c_n) is decreasing and bounded below so $\lim_{n\to\infty} c_n = \gamma \; \exists$.

Now (a_n) is Cauchy. Given $\varepsilon > 0 \; \exists N_\varepsilon$ so $\forall n, m \geq N_\varepsilon$

$|a_n - a_m| < \frac{\varepsilon}{2} \implies a_m < a_n + \varepsilon/2$

Letting $m = n, n+1, \ldots$ implies $c_n = \sup\{a_n, a_{n+1}, \ldots\} \leq a_n + \frac{\varepsilon}{2}$.

$\implies c_n \leq a_n + \varepsilon/2$ (1)

Similarly $a_n < a_m + \frac{\varepsilon}{2} \implies a_n - \varepsilon/2 < a_m \Rightarrow a_n \leq b_n + \frac{\varepsilon}{2}$ (2)

Hence (1) + (2) $\Rightarrow$ $c_n - b_n \leq \varepsilon$. Since $b_n \leq c_n$ & the inequality holds $\forall \varepsilon > 0$ we get $\lim_{n\to\infty} c_n = \gamma = \lim_{n\to\infty} b_n = \beta$.

Final step: We have $b_n \leq a_n \leq c_n$. Let $\varepsilon > 0$ be given

Define $\alpha := \beta = \gamma$. $\exists N_1$ so $|b_n - \alpha| < \varepsilon \quad \forall n \geq N_1$

$\exists N_2$ so $|c_n - \alpha| < \varepsilon \quad \forall n \geq N_1$

Thus $\forall n \geq N_\varepsilon = \max\{N_1, N_2\}$
$\alpha - \varepsilon < b_n < \alpha + \varepsilon$
$b_n \leq a_n \leq c_n$
$\alpha - \varepsilon \leq c_n < \alpha + \varepsilon$

$$\left.\begin{array}{l} \Rightarrow \alpha - \varepsilon + b_n < a_n + b_n \\ \& \quad a_n + c_n < \alpha + \varepsilon + c_n \end{array}\right\} \Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$$

$\Rightarrow |a_n - \alpha| < \varepsilon$. //