LECTURE 22: LINEAR MAPS

Theorem 42: Let X and Y be normed spaces and let $T : X \to Y$ be linear. Then T is continuous at $x = 0 \iff T$ is uniformly continuous on X.

Proof: ($\Leftarrow$) Immediate. ($\Rightarrow$) Given $\varepsilon > 0$, $\exists \delta > 0$ such that if $\|x - 0\| < \delta$ then $\|T(x) - T(0)\| < \varepsilon$. Since $T(0) = 0$, if $\|x\| < \delta$ then $\|T(x)\| < \varepsilon$. Hence if $\|x - y\| < \delta$ then $\|T(x) - T(y)\| = \|T(x - y)\| < \varepsilon$ and, since $\delta = \delta(0,\varepsilon)$ is independent of x and y, this shows that T is uniformly continuous on X.

Definition: If $T : X \to Y$ is linear we say T is bounded if $\exists\, M \geqslant 0$ such that

$$\|T(x)\|_Y \leqslant M\|x\|_X \qquad \forall\, x \in X.$$

Theorem 43: T is bounded $\iff$ T is continuous.

Proof: ($\Longrightarrow$) Given $\varepsilon > 0$ let $\delta = \varepsilon/(M+1)$. Then $\|x\| < \delta \Rightarrow$ $\|T(x)\| < M.\delta = \frac{\varepsilon.M}{M+1} < \varepsilon$.

($\Longleftarrow$) Let $\varepsilon = 1$. Then $\exists\, \delta > 0$ such that $\|x\| = \delta \Rightarrow$ $\|T(x)\| \leqslant 1$.

If $x \neq 0$ then $\|T(x)\| = \|T(\delta.x/\|x\|)\|\cdot\|x\|/\delta$. Since $\|\delta x/\|x\|\| = \delta$ RHS $\leqslant 1.\|x\|/\delta$. Let $M = 1/\delta$. Then $\forall\, x$, $\|T(x)\| \leqslant M\|x\|$.

Definition: $B(X,Y)$ = set of all bounded linear transformations from X to Y.

Definition: if $T \in B(X,Y)$ let

$$\|T\| = \inf\{M : \|T(x)\| \leqslant M\|x\| \qquad \forall\, x \in X\}$$

If $T, S \in B(X,Y)$ and $\alpha \in \mathbb{R}$ let $(T + S)(x) = T(x) + S(x)$ and $(\alpha T)(x) = \alpha.T(x)$.

Definition: A normed space is called a Banach space if it is complete.

Theorem 44: $B(X,Y)$ is a normed space which is Banach if Y is Banach.

<u>Proof</u>: We first need to prove that $\|T\|$ as defined above is a norm on $B(X,Y)$. Clearly $\|T\| \geq 0$. If $\|T\| = 0$ then $\forall\, x \in X$ $\|T(x)\| \leq \varepsilon\, \|x\|$ $\forall\, \varepsilon > 0$. Hence, fixing x, $\|T(x)\| = 0 \Rightarrow T(x) = 0 \Rightarrow$ (letting x vary) T is the zero transformation, 0. Next

$$\begin{aligned}\|(T + S)(x)\| = \|T(x) + S(x)\| &\leq \|T(x)\| + \|S(x)\| \\ &\leq \|T\|\cdot\|x\| + \|S\|\cdot\|x\| \\ &= (\|T\| + \|S\|)\cdot\|x\|\end{aligned}$$

Therefore $\|T + S\| \leq \|T\| + \|S\|$.

The proof that $\|\alpha T\| = |\alpha|\,\|T\|$ is left to the reader as are the easily checked details that $B(X,Y)$ is a vector space.

Completeness: Let Y be complete and let $\{T_n\}$ be Cauchy in $B(X,Y)$. Then $\exists\, M \geq 0$ such that $\|T_n\| \leq M$ $\forall\, n \in \mathbb{N}$. Therefore $\forall\, x \in X$ $\|T_n(x)\| \leq \|T_n\|\cdot\|x\| \leq M\|x\|$ - (1). Given $\varepsilon > 0$ $\exists\, N_\varepsilon$ such that $\|T_n - T_m\| < \varepsilon$ $\forall\, n,m \geq N_\varepsilon$. Hence $\forall\, x \in X$ $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$$\leq \varepsilon\, \|x\| \quad - (2)$$

Therefore $\{T_n(x)\}$ with x fixed is Cauchy in Y. Since Y is complete, $\lim_{n\to\infty} T_n(x)$ exists. Denote this x dependent limit by $T(x)$

The continuity of the $\|\cdot\|$ for $+$ and $\cdot$ $\Rightarrow$ T is linear since each T_n is :$[T(x) + T(y) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) = \lim_{n\to\infty}(T_n(x) + T_n(y))$

$$\begin{aligned}&= \lim_{n\to\infty}(T_n(x + y)) \\ &= T(x + y)]\end{aligned}$$

By equation (1) $\|T(x)\| \leq M\|x\|$ $\forall\, x \in X$. Therefore $T \in B(X,Y)$.

Fix x and let $m \to \infty$ in equation (2).

$$\forall\, n \geq N_\varepsilon \qquad \|T_n(x) - T(x)\| \leq \varepsilon\|x\|$$

$$\|$$

$$\|(T_n - T)(x)\| \qquad \forall\, x \in X.$$

Hence $\|T_n - T\| \leq \varepsilon \quad \forall\, n \geq N_\varepsilon$. Therefore $\lim_{n\to\infty} T_n = T$ and the Cauchy sequence converges. Thus $B(X,Y)$ is complete.

Definition: If $Y = \mathbb{R}$ or $\mathbb{C}$ when X is a real or complex vector space respectively we denote $B(X,Y)$ by X' and call the space the <u>topological dual</u> or <u>adjoint</u> of X.

<u>Exercises</u> (i) If N is a real normed vector space and $F \in N'$ then $\|F\| = \sup\{F(x) : \|x\| = 1\}$.

<u>Exercise</u> (ii) The topological dual of any normed vector space is complete.

<u>Exercise</u> (iii) Let N be the set of polynomials on $[0,1]$. If $p(t) = a_0 + a_1 t + \ldots + a_n t^n \in N$ let $\|p\| = |a_0| + \ldots + |a_n|$. Show that N is normed but not Banach. Find N'.

<u>Exercise</u> (iv) $(\mathbb{R}^n)' \cong \mathbb{R}^n$; any linear map $T : \mathbb{R}^n \to \mathbb{R}$ is continuous.

<u>Exercise</u> (v) Show that the set of all real bounded sequences can be given a Banach space norm.

"The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell."

(St. Augustine)