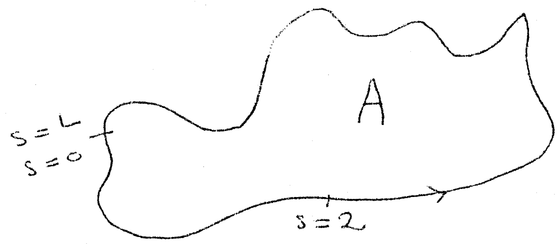
LECTURE 30: ISOPERIMETRIC THEOREM

Theorem 65: Among all simple, closed, piecewise smooth curves in $\mathbb{R}^2$ of length L the circle encloses the maximum area.



Proof: Let s be the arc length parameter $0 \leq s \leq L$ and let A be the area enclosed by the curve $(x(s), y(s))$. Let $t = 2\pi s/L$; then $0 \leq t \leq 2\pi$. We will use the $c.o.n.$ set in $L_2[0,2\pi]$ given by

$$\{1/\sqrt{2\pi},\ (1/\sqrt{\pi})\cos nt,\ (1/\sqrt{\pi})\sin nt : n \in \mathbb{N}\}$$

and express $x(t)$ and $y(t)$ in their fourier series forms:

$$x(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

$$y(t) = b_0/2 + \sum_{n=1}^{\infty} (c_n \cos(nt) + d_n \sin(nt)) \ .$$

Assumption: We may differentiate each series term by term and obtain the series for the derivative in each case. [Recall that convergence in $H = L_2$ does not necessarily mean even pointwise convergence.] This assumption needs justification which is not given here. Then

$$\frac{dx}{dt} = 0 + \sum_{n=1}^{\infty} (nb_n \cos(nt) + (-na_n)\sin(nt))$$

$$\frac{dy}{dt} = 0 + \sum_{n=1}^{\infty} (nd_n \cos(nt) + (-nc_n)\sin(nt))$$

Since $ds^2 = dx^2 + dy^2$

[or, what is more precise $s = \int \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt$] .

We have $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$. But

$$\frac{dx}{ds} = \frac{dx}{dt} \cdot \frac{dt}{ds} = \frac{dx}{dt} \cdot \frac{2\pi}{L} \quad \text{and} \quad \frac{dy}{ds} = \frac{dy}{dt} \cdot \frac{2\pi}{L} \quad \Rightarrow$$

$$\left(\frac{dx}{dt}\right)^2 \left(\frac{2\pi}{L}\right)^2 + \left(\frac{dy}{dt}\right)^2 \left(\frac{2\pi}{L}\right)^2 = 1 \quad \Rightarrow$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{L}{2\pi}\right)^2 .$$

Therefore

$$2\pi\left(\frac{L}{2\pi}\right)^2 = \int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt = \left\|\frac{dx}{dt}\right\|^2 + \left\|\frac{dy}{dt}\right\|^2$$

$$= \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2 + c_n^2 + d_n^2)$$

We now invoke Green's Theorem for $\mathbb{R}^2$, namely that

$$\oint_{\text{curve}} N \, dy = \iint_{\substack{\text{region} \\ \text{enclosed}}} \frac{\partial N}{\partial x} \, dxdy$$

with $N(x,y) = x$. From this $\partial N/\partial x = 1$ and so

$$A = \iint 1 \, dxdy = \oint x \, dy$$

$$= \int_0^{2\pi} x(t) \frac{dy}{dt} \, dt$$

$$= (x, \frac{dy}{dt})$$

$$= \sum_{n=1}^{\infty} (n\text{th coef } x) \cdot (n \text{ coef } \frac{dy}{dt})$$

$$= \sum_{n=1}^{\infty} (na_n d_n - nb_n c_n)$$

Consider

$$L^2 - 4\pi A \text{ : since}$$

$$L^2 = 2\pi\left[\sum_{n=1}^{\infty} n^2(a_n^{\ 2} + b_n^{\ 2} + c_n^{\ 2} + d_n^{\ 2})\right] \quad \text{and}$$

$$4\pi A = 4\pi\left[\sum_{n=1}^{\infty} (na_n d_n - nb_n c_n)\right] \quad \text{then}$$

$$L^2 - 4\pi A = 2\pi\left[\sum_{n=1}^{\infty}\{(na_n - d_n)^2 + (nb_n + c_n)^2 + (n^2 - 1)(c_n^{\ 2} + d_n^{\ 2})\}\right] \qquad - (1)$$

[This equation (1) needs to be checked] Hence

$$L^2 - 4\pi A \geq 0 \qquad - \quad (2)$$

Note that when the curve is a circle radius R that $L = 2\pi R$ and $A = R^2$ so that

$$L^2 - 4\pi A = 4\pi^2 R^2 - 4\pi\pi R^2$$
$$= 0 .$$

So that for fixed L the maximum area compatible with the necessary condition (2) is attained by a circle. It remains to prove that this is the only shape enclosing maximum area.

If $L^2 - 4\pi A = 0$ so must each term on the RHS of (1) i.e. $\forall\, n \in \mathbb{N}$

$$na_n - d_n = 0 \text{ and } nb_n + c_n = 0 \text{ and}$$

$$\forall\, n > 1,\ c_n = 0 \text{ and } d_n = 0$$

$$n = 1 \Rightarrow a_1 = d_1 \text{ and } b_1 = -c_1$$

$$n > 1 \Rightarrow a_n = 0 \text{ and } b_n = 0$$

$$c_n = 0 \text{ and } d_n = 0$$

Thus

$$x(t) = a_0/2 + a_1\cos(t) + b_1\sin(t) + 0$$
$$y(t) = c_0/2 + -b_1\cos(t) + a_1\sin(t) + 0$$

i.e. $(x - a_0/2)^2 + (y - c_0/2)^2 = a_1^{\ 2} + b_1^{\ 2}$, which is the equation of a circle.