LECTURE 28: THE PROJECTION THEOREM

Theorem 57: Let $K \subset H$ be a closed convex subset of a hilbert space H. Then if $x_0 \in H\setminus K$, there exists a unique element $y_0 \in K$ with

$$\|x_0 - y_0\| = \inf\{\|x_0 - y\| \; y \in K\} = d.$$

That is, there exists a point y_0 in K closest to x_0.

Proof: (1) Existence

There exists a sequence of points (y_n) in K with

$$d = \lim_{n\to\infty} \|x_0 - y_n\|$$

Now $$\|y_m - y_n\|^2 = 2\|y_m - x_0\|^2 + 2\|y_n - x_0\|^2 - \|(y_m - x_0) + (y_n - x_0)\|^2$$

by applying the parallelogram law to the pair

$$y_m - x_0, \quad y_n - x_0.$$

Now $$\|y_m - x_0 + (y_n - x_0)\|^2 = \|y_m + y_n - 2x_0\|^2 = 4\left\|\frac{y_m + y_n}{2} - x_0\right\|^2 \geq 4d^2 .$$

This follows since K is assumed convex $\Rightarrow \dfrac{y_m + y_n}{2} \in K$ for $y_m, y_n \in K$.

Thus $$\|y_m - y_n\|^2 \leq 2\|y_m - x_0\|^2 + 2\|y_n - x_0\|^2 - 4d^2$$

the whole expression on the RHS tends to zero as $m, n \to \infty$.

Thus (y_n) is a Cauchy sequence in K. Since H is complete (y_n) is convergent $y_n \to y_0$ in H, K is closed $\Rightarrow y_0 \in K$ and

$$d = \lim_{n\to\infty} \|x_0 - y_n\| = \|x_0 - y_0\|$$ since the norm is continuous.

<u>Uniqueness</u> of y_0 : let $z_0 \in K$ satisfy $d = \|x_0 - z_0\|$ also. Then
$\|y_0 - z_0\|^2 = 2\|y_0 - x_0\| + 2\|z_0 - x_0\|^2 - \|(y_0 - x_0) + (z_0 - x_0)\|^2$

The last term is just

$$4\left\| \frac{y_0+z_0}{2} - x_0 \right\|^2 \leq 4d^2 \text{ since } \frac{y_0 + z_0}{2} \in K.$$

So the entire expression is $\leq$

$$2d^2 + 2d^2 - 4d^2 = 0 \Rightarrow \|y_0 - z_0\| = 0 \Rightarrow y_0 = z_0 \ .$$

Hence the point y_0 is unique in K.

Let M be a closed subspace of H.

Let $M^\perp = \{y \in H : y \perp x \quad \forall\, x \in M\}$

Then $M^\perp$ is a closed subspace of H : $M^\perp$ is closed for if $(y_n) \subset M^\perp$ and $y_n \to y$ in H then if $x \in M$

$$(y_n, x) = 0 \Rightarrow \qquad (y, x) = \lim_{n\to\infty} (y_n, x) = 0$$

$$\Rightarrow y \in M^\perp \ .$$

Now $H = M \oplus M^\perp$ since each $x_0 \in H$ is given by

$$x_0 = \underset{\underset{M^\perp}{\cup}}{(x_0 - y_0)} + \underset{\underset{M}{\cup}}{y_0}$$

and $M \cap M^\perp = \{0\}$.

<u>Theorem 58</u>: $x_0 - y_0 \in M^\perp$.

Proof: Let $y \in M$, $a \in \mathbb{C} \Rightarrow y_0 + ay \in M$

$$\|x_0 - (ay + y_0)\|^2 \geq \|y_0 - x_0\|^2$$

$$\Rightarrow -\bar{a}(x_0 - y_0, y) - a(y, x_0 - y_0) + |a|^2 \|y\|^2 \geq 0 .$$

Let $a = \varepsilon > 0$

$$\varepsilon(x_0 - y_0, y) + \varepsilon(y, x_0 - y_0) \leq \varepsilon^2 \|y\|^2 \Rightarrow 2\mathrm{Re}(x_0 - y_0, y) \leq 0.$$

Then set $a = i\varepsilon$, $\varepsilon > 0$

and obtain

$$2\mathrm{Im}(x_0 - y_0, y) \leq \varepsilon\|y\|^2 \qquad \Rightarrow 2\mathrm{Im}(x_0 - y_0, y) \leq 0.$$

Replace y by $-y$ and use same argument as above:

$$-2\mathrm{Re}(x_0 - y_0, y) \leq 0$$

$$-2\mathrm{Im}(x_0 - y_0, y) \leq 0$$

$$\Rightarrow \qquad (x_0 - y_0, y) = 0 \Rightarrow x_0 - y_0 \in M^{\perp} .$$

Projection maps:

Given $H = M \oplus M^{\perp}$ define a projection map

$$p : H \to M \text{ by } p(x_0) = y_0 \text{ where } x_0 = (x_0 - y_0) + y_0$$

Theorem 59:

Properties of the projection map p

(1) p is linear

(2) p is idempotent, $p^2 = p$

(3) bounded, $\|p\| = 1$

(4) p is onto M

(5) $M^{\perp} = (I - p)(H)$.

Proofs: Exercise.

Theorem 60: H is isomorphic to H' = (set of bounded linear functions).

<u>Proof</u>: $|(x_0,y_0)| \leq \|x_0\|\ \|y_0\|$

Define $F_{y_0}(x) = (x,y_0) \in \mathbb{C}$.

F_{y_0} is linear and

$$\|F_{y_0}\| \leq \|y_0\| \Rightarrow F_{y_0} \text{ is bounded. Let } y_0 \neq 0.$$

$$F_{y_0}\left(y_0/\|y_0\|\right) = \left(y_0/\|y_0\|, y_0\right) = \|y_0\|^2/\|y_0\| = \|y_0\|$$

By alternate definition of $\|F\|$: $\|F_{y_0}\| \geq \|y_0\| \Rightarrow \|F_{y_0}\| = \|y_0\|$.

Define $\varphi : H \to H', \varphi(y_0) = F_{y_0}$; $\varphi(ax + by) = \bar{a}\varphi(x) + \bar{b}\ \varphi(y)$.

<u>Claim</u> φ is an onto map:

Let $F \in H'$ we will prove that $\exists !\ y_0 \in H : F(x) = (x,y_0)$. $F = 0$ is the trivial case.

$F \neq 0$. Let $N = \ker F = F^{-1}(0)$. F is continuous $\Rightarrow$ N is closed, F linear $\Rightarrow$ N is a subspace, $F \neq 0 \Rightarrow$ N is a proper subspace.

Let $x_0 \in N^{\perp}$ and $y_0 = ax_0$ be chosen so that

$$F(x_0) = (x_0,y_0) = (x_0,ax_0) = \bar{a}\ \|x_0\|^2 . \text{ That is}$$

$$a = \frac{F(x_0)}{\|x_0\|^2} .$$

Let $x \in H$ and $b = F(x)/F(x_0)$ then $F(x - bx_0) = F(x) - \frac{F(x)}{F(x_0)}\ F(x_0)$

$$= 0$$

Thus $(x,y_0) = (x - bx_0,y_0) + (bx_0,y_0)$ since $y_0 \in N^{\perp}$

$$= b(x_0,y_0) = \frac{F(x)}{F(x_0)} \cdot F(x_0) = F(x).$$

Therefore φ is onto and normpreserving. Hence H and H' are (nearly) linearly isometric.