LECTURE 27: HILBERT SPACES

Let H be a real or complex vector space. Then $(\ ,\ )$ is an inner product on $H \times H$ iff

(i) $(x,y) = \overline{(y,x)}$

(ii) $(ax_1 + bx_2, y) = a(x_1,y) + b(x_2,y)$

(iii) $(x,x) \geq 0$, $(x,x) = 0 \Rightarrow x = 0$.

Define $\|x\| = (x,x)^{\frac{1}{2}}$. Then this defines a norm on H. H will be called a Hilbert space if the topology generated by $\|\cdot\|$ is complete.

Theorem 51: [The Cauchy-Schwarz inequality.]

$|(x,y)| \leq \|x\|\cdot\|y\|$ for all x, $y \in H$.

Proof: We may assume both x and y are non zero elements for in case one is zero the result holds trivially.

Let $z = ax + by$ then $(z,z) \geq 0 \Rightarrow$

$$(ax + by,\ ax + by) = (ax,ax) + (ax,by) + (by,ax) + (by,by)$$
$$= a\bar{a}(x,x) + a\bar{b}(x,y) + \bar{a}b(y,x) + b\bar{b}(y,y) \geq 0\ .$$

Take $a = (y,y)$ and cancelling the positive factor (y,y) in the inequality we then obtain

$$(y,y)(x,x) + \bar{b}(x,y) + b(y,x) + b\bar{b} \geq 0.$$

Now choose $b = -(x,y)$ and so $\bar{b} = \overline{-(x,y)} = -(y,x)$ and so the inequality becomes

$$(y,y)(x,x) - (y,x)(x,y) - (x,y)(y,x) + (x,y)(y,x) \geq 0$$

$\Rightarrow$ $(y,y)(x,x) \geq (x,y)(y,x) = |(x,y)|^2$.

$\Rightarrow$ $|(x,y)| \leq \|y\|\ \|x\|$.

Theorem 52: $\|\cdot\|$ is in fact a norm on H.

Proof: (1) $\|ax\| = (ax,ax)^{\frac{1}{2}} = \{a\bar{a}(x,x)\}^{\frac{1}{2}}$

$$= \{|a|^2(x,x)\}^{\frac{1}{2}} = |a|(x,x)^{\frac{1}{2}} = |a|\ \|x\|.$$

(2) $\|x + y\| \leq \|x\| + \|y\|$

$$\|x + y\|^2 = (x + y, x + y) = \underbrace{(x,x)}_{\|x\|^2} + \underbrace{(y,y)}_{\|y\|^2} + \underbrace{(x,y) + (y,x)}_{2\mathrm{Re}(x,y)}$$

Now $2\mathrm{Re}(x,y) \leq 2\,|(x,y)|$

Now by the Cauchy Schwarz inequality

$$2|(x,y)| \leq 2\|x\|\cdot\|y\|$$

Therefore $\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\ \|y\| + \|y\|^2 = \{\|x\| + \|y\|\}^2$

Thus $\|x + y\| \leq \|x\| + \|y\|$.

Examples of Hilbert Spaces.

(1) $l_2(\mathbb{R}) = \{(x_i) : x_i \in \mathbb{R}$ and $\sum_i x_i^2 < \infty\}$

Define $(x,y) = \sum_i x_i y_i$

$l_2(\mathbb{C})$ same as above but with $(x,y) = \sum_i x_i \bar{y}_i$

(2) $L_2[0,1]$ = set of Lebesque measurable "functions" on $[0,1]$. Satisfying

$$\int_0^1 |f|^2 < \infty$$

$(x,y) = \int x(t)\bar{y}(t)dt \qquad x : [0,1] \to \mathbb{C}$.

Theorem 53: $(\ ,\ ) : H \times H \to \mathbb{C}$ defines a continuous function.

Proof: We will show $\lim_{n\to\infty}(x_n,y_n) = (x,y)$ for all sequences $(x_n) \to x$, $(y_n) \to y$.

$$|(x_n,y_n) - (x,y)| = |(x_n,y_n) - (x,y_n) + (x,y_n) - (x,y)|$$

$$\leq |(x_n,y_n) - (x,y_n)| + |(x,y_n) - (x,y)|$$

$$= |(x_n - x,y_n)| + |(x,y_n - y)|$$

$$\leq \|x_n - x\| \; \|y_n\| + \|x\| \; \|y_n - y\| \;..$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 0 & \leq m & 0 \end{array}$$

since $\|y_n\| \leqq m$ for some m.

Definition: Let us write $x \perp y$ if $(x,y) = 0$ and if this is so say that x is perpendicular to y.

Theorem 54: Pythagoras' Theorem.

If $x \perp y$ then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Theorem 55: The parallelogram law: Real case.

$$\forall \; x, \; y \in H \qquad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \; .$$

Proof: Exercise.

Theorem 56: If $(B, \|\bullet\|)$ is a banach space and $\|\bullet\|$ satisfies the parallelogram law, then the expression

(*) $$(x,y) = \tfrac{1}{4}\{\|x + y\|^2 - \|x - y\|^2\}$$

defines an inner product space with the same structure such that

$$(x,x)^{\frac{1}{2}} = \|x\| \; .$$

Proof: We must verify the properties of an inner product for expression (*)

(1) $(x,x) = \tfrac{1}{4}\{2^2\|x\|^2 + \|0\|^2\} = \; = \|x\|^2 \geq 0.$

(2) $\|u + v + w\|^2 + \|u + v - w\|^2$

$= 2\|u + v\|^2 + 2\|w\|^2$

Also $\|u - v + w\|^2 + \|u - v - w\|^2 = 2\|u - v\|^2 + 2\|w\|^2$

Subtracting yields

$$\{\|u + w + v\|^2 - \|u + w - v\|^2\} + \|u - w + v\|^2$$
$$- \|u - w - v\|^2 = 2(\|u + v\|^2 - \|u - v\|^2)$$

Now applying the definition of (,) we obtain

$$4(u + w,v) + 4(u - w,v) = 8(u,v).$$

Let $u = w$

$$(u + u,v) + (u - u,v) = 2(u,v)$$

$$\|$$

$$0$$

$\Rightarrow$ $$(2u,v) = 2(u,v)\ .$$

Let $x_1 = u + w,\ x_2 = u - w,\ y = v \Rightarrow$

$$(x_1,y) + (x_2,y) = 2(u,v) = (2u,v) = (x_1 + x_2,y)\ .$$

Prove $(au,v) = a(u,v)$ as an exercise.

This completes the proof. There is an identity corresponding to (*) in the case of a complex banach space but it is much more complicated and will not be given here.

"One could say that application's constant relation to theory is the same as that of the leaf to the tree: one supports the other, but the former feeds the latter."

(Hadamand)