Chapter 7: Properties of differentiable functions

Theorem: (Rolle's Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If $f(a) = f(b)$ and f is differentiable on (a,b) then there exists a point $x_0 \in (a,b)$ such that $f'(x_0) = 0$.*

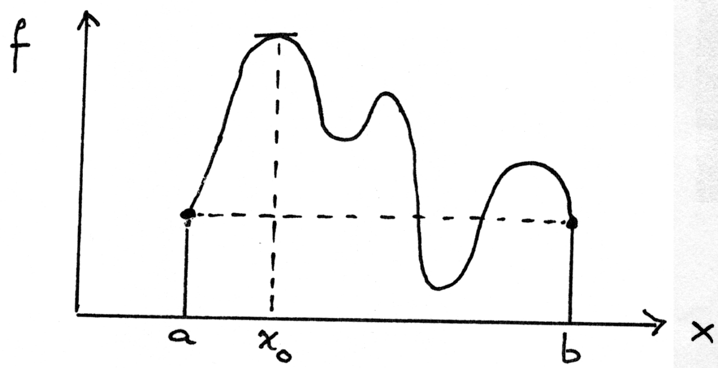

Figure 18: Rolle's Theorem.

Proof:

1. If f is constant on $[a,b]$ then $f'(x) \equiv 0$ on (a,b). Hence we may suppose that f is not constant on $[a,b]$.

2. Therefore, either (i) $\max\{f(x) : x \in [a,b]\} > f(a) = f(b)$ or (ii) $\min\{f(x) : x \in [a,b]\} < f(a) = f(b)$. We shall only consider case (i) as case (ii) is similar.

3. Choose $x_0 \in (a,b)$ such that $f(x_0) = \sup\{f(x) : x \in [a,b]\}$. Then since $f(x) \leqslant f(x_0)$ for all $x \in [a,b]$, $(f(x) - f(x_0))/(x - x_0) \leqslant 0$ for all $x \in [a,b]$ with $x_0 < x$.

4. Therefore, $f'_+(x_0) = \lim_{x \to x_0^+} (f(x) - f(x_0))/(x - x_0) \leqslant 0$.

5. On the other hand, $f(x) - f(x_0) \leqslant 0$ for all $x \in [a,b]$ with $x < x_0$ and so $(f(x) - f(x_0))/(x - x_0) \geqslant 0$ for all $x \in [a,b]$ with $x < x_0$.

6. Therefore, $f'_-(x_0) = \lim_{x \to x_0^-} (f(x) - f(x_0))/(x - x_0) \geqslant 0$.

7. Now since f is differentiable at x_0, $f'_+(x_0) = f'_-(x_0)$ and so

$$f'(x_0) = f'_+(x_0) = f'_-(x_0) = 0.$$

□

Exercise: Let $f : [-1,1] \to \mathbb{R}$ be defined by, $f(x) := (x+1)^m (x-1)^n$. Show that $f'(x_0) = 0$, where $x_0 := (m-n)/(m+n)$.

Example: Consider the function $f : [-1,1] \to \mathbb{R}$ defined by, $f(x) := |x|$. Show that $f'(x)$ never equals 0. Does this contradict Rolle's theorem?

Answer: First, note that $f'(0)$ does not exist and $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Therefore, $f'(x)$ never equals 0. However, this does not contradict Rolle's theorem,

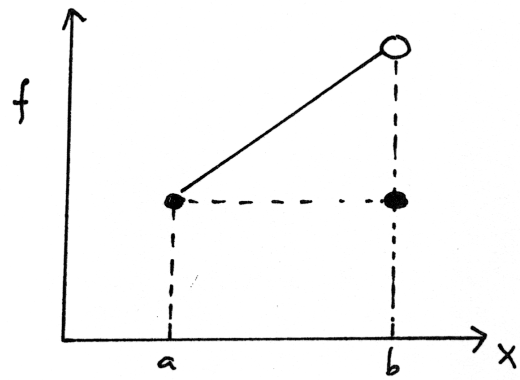

Figure 19: Continuity on [a,b] is needed for Rolle's Theorem.

because although f is continuous on $[-1,1]$ and $f(-1) = f(1)$, f is **not** differentiable on $(-1,1)$. $\square$

Theorem: (Mean Value Theorem) *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If f is differentiable on (a,b) then there a point $x_0 \in (a,b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

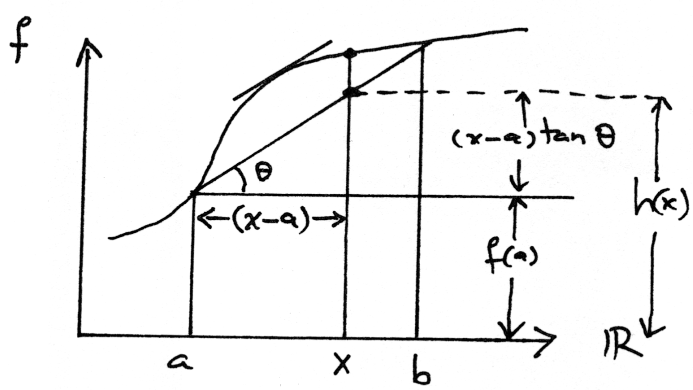

Figure 20: Mean Value Theorem .

Proof:

1. Let $h : [a,b] \to \mathbb{R}$ be the equation of the line joining the points $(a, f(a))$ and $(b, f(b))$, ie:

$$h(x) := \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

2. Now let us consider the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) := f(x) - h(x)$, ie: $g(x)$ is the difference between $f(x)$ and the line described by h.

3. Then g is continuous on $[a, b]$, differentiable on (a, b) and $g(a) = g(b)$.

4. Therefore by Rolle's theorem there exists a point $x_0 \in (a, b)$ such that $g'(x_0) = f'(x_0) - h'(x_0) = 0$, ie: $f'(x_0) = h'(x_0) = \dfrac{f(b) - f(a)}{b - a}$. $\square$

Example: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that f is increasing if, and only if, $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$.

Answer: Suppose that $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$.

Let a, b be any real numbers such that $a < b$; we need to show that $f(a) \leqslant f(b)$.

First we note that f is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by the mean value theorem there exists a point $x_0 \in (a, b)$ such that

$$f(b) - f(a) = f'(x_0) \cdot (b - a) \geqslant 0.$$

From this it follows that $f(a) \leqslant f(b)$.

Conversely, suppose that f is increasing on $\mathbb{R}$. Let x_0 be any point in $\mathbb{R}$ and let $x_0 < x$. Then $f(x) \geqslant f(x_0)$, ie: $f(x) - f(x_0) \geqslant 0$. Therefore,

$$\frac{f(x) - f(x_0)}{x - x_0} \geqslant 0$$

for all $x > x_0$ and so

$$f'(x_0) = f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \geqslant 0$$

. $\square$

Exercise: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that f is strictly increasing if $f'(x) > 0$ for all $x \in \mathbb{R}$.

We now show that the derivative of an (everywhere) differentiable function satisfies an intermediate value property similar to that satisfied by continuous functions, despite the fact that f' may not be continuous.

Lemma: *Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is differentiable at both a and b. If,*

(i) $f'(a) > 0$ then there exists a $\delta > 0$ such that $f(x) > f(a)$ for all $x \in [a, b]$ with $a < x < a + \delta$;

(ii) $f'(b) < 0$ then there exists a $\delta > 0$ such that $f(x) > f(b)$ for all $x \in [a, b]$ with $b - \delta < x < b$.

Proof: We shall only consider case (i) as case (ii) is similar. Since $f'(a) > 0$ there exists a $\delta > 0$ such that $(f(x) - f(a))/(x - a) > 0$ for all $a < x < a + \delta$. Therefore, for all $x \in [a, b]$ with $a < x < a + \delta$,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) > 0.$$

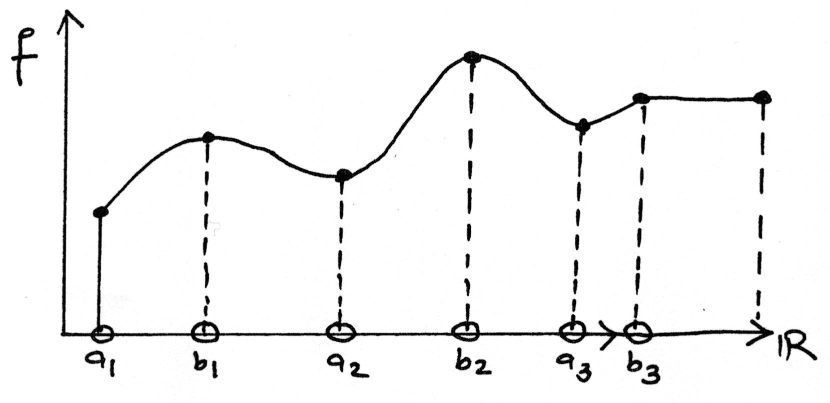

Figure 21: Positive derivative means increasing.

□

Theorem: (Darboux's Theorem) *Suppose that $a < b$ and $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$. If $f'(a) > k > f'(b)$ (or vice versa) then there exists a point $x_0 \in (a, b)$ such that $f'(x_0) = k$.*

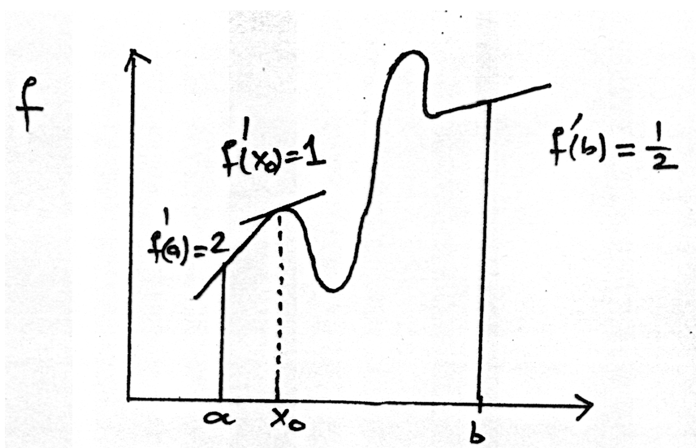

Figure 22: Darboux's Theorem.

Proof: 1. Suppose that $f'(a) > k > f'(b)$ and consider the function $g : [a, b] \to \mathbb{R}$ defined by, $g(x) := f(x) - kx$.

2. Since g is continuous on $[a, b]$, g attains its maximum value at some point $x_0 \in [a, b]$.

3. Moreover, since $g'(a) > 0$ and $g'(b) < 0$ it follows from the previous Lemma that $x_0 \in (a, b)$.

4. The result now follows as in Rolle's theorem by showing that $g'(x_0) = 0$. The proof for the case $f'(a) < k < f'(b)$ is similar. □