

**Aside: Formal definition of a limit**

We have said that

$$\lim_{x \rightarrow a} f(x) = L$$

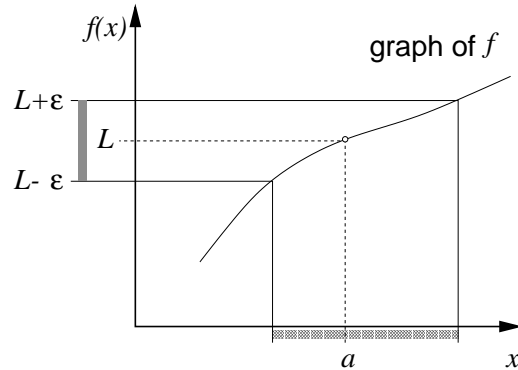
means that we can make  $f(x)$  arbitrarily close to  $L$  by choosing  $x$  to be close enough to  $a$ .

Need to be precise about what “arbitrarily close to  $L$ ” means.

“Arbitrarily close” means that if you pick any very small number—call it  $\epsilon$ —then we must be able to choose  $x$  such that  $f(x)$  is within  $\epsilon$  of  $L$ . That is,

$$L - \epsilon < f(x) < L + \epsilon.$$

On the diagram below, this means that  $f(x)$  must be in the grey shaded region.



Any  $x$  which starts in the cross-hatched region will be such that  $f(x)$  is within  $\epsilon$  of  $L$ —this is our set of points which are “close enough to  $a$ ”. For simplicity, we would like to describe this set in a nice way. Clearly, there exists a small positive  $\delta$  such that the interval

$$(a - \delta, a + \delta)$$

is contained in the cross-hatched region. Therefore, we have

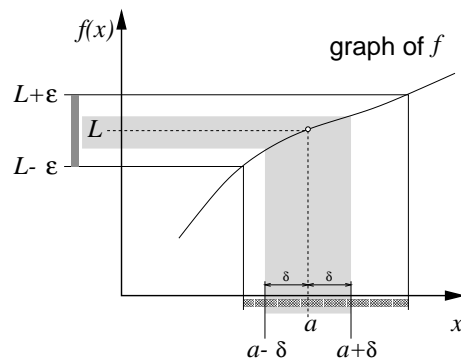
$$x \in (a - \delta, a + \delta) \implies f(x) \in (L - \epsilon, L + \epsilon).$$

The idea is illustrated in the diagram below. To make a formal definition of a limit we just write all of this down in the right order:

**Definition of limit.** We have

$$\lim_{x \rightarrow a} f(x) = L$$

if given any  $\epsilon > 0$  it is possible to find a  $\delta > 0$  such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon.$$


**Note:** Although we make this formal definition, it is not always easy to find a  $\delta$  for the given  $\epsilon$ . That is to say, the definition is difficult to work with! It is used primarily to prove theorems which allow you to avoid cumbersome processes of limit evaluation. This approach is beyond the scope of this paper, although it underlies all of our work. □