Infinite limits

Example. In studying

$$\lim_{x\to 3}\frac{1}{(x-3)^2}$$

we saw that the closer x got to 3, the larger the value of $1/(x-3)^2$ became. We thus defined the value of the limit to be "infinity". □

> **Definition.** We say that
>
> $$\lim_{x\to a} f(x) = \infty$$
>
> if $f(x)$ becomes arbitrarily large as x approaches a.

Here we have implicitly defined the symbol "∞" as something which is "arbitrarily large". In fact, the intention is to define infinity as something which is **larger than every real number**. Thus, we say that $f(x) \to \infty$ as $x \to a$ if we can show that given *any real number* A we can make

$$f(x) > A$$

by choosing x to be close enough to a.

> **Exercise.** Make a formal definition of the statement
>
> $$\lim_{x\to a} f(x) = -\infty.$$
>
> (Hint: think about what "$-\infty$" is.) □

Arithmetic with Infinity

Some limit calculations are simplified by the ability to do arithmetic with ∞. The allowed operations are summarized in the adjacent table (and can be proved).

	$a+\infty$	$a-\infty$	$a\cdot\infty$	a/∞
$a=-\infty$	X	$-\infty$	$-\infty$	X
$-\infty < a < 0$	$+\infty$	$-\infty$	$-\infty$	0
$a = 0$	$+\infty$	$-\infty$	X	0
$0 < a < \infty$	$+\infty$	$-\infty$	$+\infty$	0
$a = \infty$	$+\infty$	X	$+\infty$	X

An X indicates that no value can be assigned.

One–sided limits

In lectures we had an informal definition of *one-sided lim* motivated by the fact that a function can approach differ values when you come in from the left or the right.

> **(Formal) Definition of one-sided limits.**
>
> • We have
>
> $$\lim_{x\to a^+} f(x) = L$$
>
> if for every $\epsilon > 0$ there exists a $\delta_+ > 0$ such that
>
> $$a < x < a + \delta_+ \Rightarrow |f(x) - L| < \epsilon.$$
>
> We say that L is the *one-sided limit of* $f(x)$ *as* x *a proaches* a *from above* or L is the *right-hand limit*.
>
> • Similarly,
>
> $$\lim_{x\to a^-} f(x) = M$$
>
> if for every $\epsilon > 0$ there exists a $\delta_- > 0$ such that
>
> $$a - \delta_- < x < a \Rightarrow |f(x) - M| < \epsilon.$$
>
> We say that M is the *one-sided limit of* f *as* x *approach* a *from below*, or M is the *left-hand limit*.

Theorem. *We have* $\lim_{x\to a} f(x) = L$ *if and only if both left and right–hand limits of* f *exist at* a *and*

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Exercise. Prove this theorem. (Hint: to show that existence of the limit implies the existence and equality one–sided limits is easy. For the other way, suppose b one–sided limits exist, and let $\delta = \min\{\delta_-, \delta_+\}$ in the vari definitions.)