

Let f and g be functions which are differentiable at some point(s) x_0 . Then

- Sum rule.

$$(f + g)' = f' + g'$$

- Product rule.

$$(f \times g)' = f'g + fg'$$

- Quotient rule.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{[g]^2} \quad [\text{if } g(x_0) \neq 0]$$

- Power rule. If $f(x) = x^n$ for any integer n , then

$$f'(x) = nx^{n-1}.$$

- Chain rule. If $y = f(u)$ and $u = g(x)$, then $y = f(g(x)) \equiv y(u(x))$ and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = f'(u)g'(x).$$

EXERCISE: write these rules in the $\frac{d}{dx}$ notation.

*Proof of the sum rule:*¹

$$\begin{aligned} (f + g)'(x) &\stackrel{\text{df}}{=} \lim_{h \rightarrow 0} \frac{(f + g)(x + h) - (f + g)(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) + g(x + h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x + h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

¹For all of these proofs we start with the definition of the derivative, and then use our rules for limits.

Proof of the product rule: As in the sum rule, we proceed by direct calculation. This time, we need a trick: we “add zero” to the limit.

$$\begin{aligned}
 (fg)'(x) &\stackrel{\text{df}}{=} \lim_{h \rightarrow 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\begin{pmatrix} f(x+h)g(x+h) & -f(x)g(x+h) \\ -f(x)g(x) & +f(x)g(x+h) \end{pmatrix}}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{(f(x+h) - f(x))g(x+h)}{h} + \frac{f(x)(g(x+h) - g(x))}{h} \right] \quad \text{[re-group]} \\
 &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) \right] + \lim_{h \rightarrow 0} \left[f(x) \frac{g(x+h) - g(x)}{h} \right] \quad \text{[sum rule (limits)]} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \lim_{h \rightarrow 0} g(x+h) + f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= f'(x)g(x) + f(x)g'(x).
 \end{aligned}$$

In this calculation we have used only rules for manipulating limits, the definition of the derivative, and the fact that for any differentiable function g ,

$$\lim_{h \rightarrow 0} g(x+h) = g(x). \quad (*)$$

Exercise. Try proving Eqn. (*), using the ϵ - δ definition of a limit.

Proof of the quotient rule: The proof is in two parts.

First of all we prove that the function $1/g$ is differentiable at x , and then we apply the product rule. For the first part, since $g(x) \neq 0$, we can use Eqn. (*) to deduce that $g(x+h) \neq 0$ for h which are small. Thus, we are allowed to divide by $g(x+h)$.

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{g(x) - g(x+h)}{g(x)g(x+h)h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{-(g(x+h) - g(x))}{h} \times \frac{1}{g(x)g(x+h)} \right] \\
 &= - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \times \lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \\
 &= -g'(x) \times \frac{1}{\lim_{h \rightarrow 0} g(x+h)g(x)} \\
 &= -\frac{g'(x)}{[g(x)]^2}.
 \end{aligned}$$

Second part: since $1/g$ is differentiable, we can use the product and chain rules as follows:

$$\begin{aligned}
 \left(\frac{f}{g}\right)' &= \left(f \frac{1}{g}\right)' = f' \frac{1}{g} + f \left(\frac{1}{g}\right)' \\
 &= f' \frac{1}{g} + f \left[-\frac{g'}{g^2}\right] = \frac{f'g - fg'}{g^2}.
 \end{aligned}$$

Exercise: Redo the second part of the proof using the $\frac{d}{dx}$ notation.