Part A. Functions, Limits, and Continuity

§0: Overview

§ 0.1 General Motivation

In this course we will study the Calculus of functions of one variable. This body of knowledge includes useful techniques for solving many problems, both abstract and practical. You will learn many of these techniques, as well as developing your mathematical reasoning skills. Primarily, we will learn about **differentiation** and **integration**, with applications to understanding the behaviour of functions of one-variable and the phenomena they model. In particular, we will study:

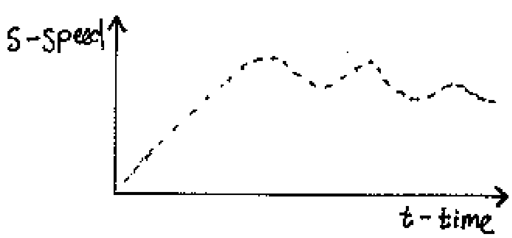
- rates of change of functions, leading to approximation and optimization
- how to compute areas and solve modelling problems involving rates of change

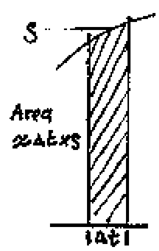
Why study calculus?

Calculus was developed (independently by Newton and Leibnitz) in the mid 17^{th} century. Newton was interested in various problems in physics for which the mathematical technology to solve them was unknown.

Ex. 1

Suppose that you know the *velocity* of some object at a sequence of times. These could be represented pictorially on a graph:





With time (denoted by t) on the horizontal axis, we can think of the velocity $v = v(t)$ as being a "function" of time. We would like to know if, using only this data, we can explain the motion of the object. For example, can we determine its acceleration. A necessary step will be to *measure* and *quantify* CHANGES in v. (We could then seek descriptions of the forces which cause these changes). This involves developing a notion of *rate of change*.

To calc the r/c of $v(t)$ we compare the value of v at two "close together" times, say t_0 and $t_0 + \Delta t$.

Δt is a small change in t.

$$\begin{aligned} \text{change in } v &= v(t_0 + \Delta t) - v(t_0) \\ \text{change in } t &= (t_0 + \Delta t) - t_0 = \Delta t \end{aligned}$$

Hence,

$$\begin{aligned}\text{rate of change of } v &\approx \frac{\text{change in } v}{\text{change in } t}\\ &= \frac{v(t_0 + \Delta\, t) - v(t_0)}{\Delta\, t}\end{aligned}$$

For example, suppose $v(t) = t^2$, then

$$\begin{aligned}\text{r/c of } v &\approx \frac{(t_0 + \Delta t)^2 - t_0^2}{\Delta t}\\ &= \frac{\left[t_0^2 + 2t_0\Delta t + (\Delta t)^2\right] - t_0^2}{\Delta t}\\ &= 2t_0 + \Delta t\\ &\approx 2t_0 \quad \text{since } \Delta t \text{ is small.}\end{aligned}$$

Making such calculations precise leads to **differential calculus**.

We may also be interested in the total distance travelled by the object. Beginning with the rough (physical) notion that

$$\text{speed} \approx \frac{\text{distance travelled}}{\text{time elapsed}}$$

we conclude that

$$\text{distance travelled} \approx \text{velocity} \times \text{time elapsed.}$$

Looking at a small time interval (Δt), we can see that **area** under the velocity-time graph for the interval Δt is approximately $\Delta t \times v(t)$, which is approximately the distance travelled between times t and $t + \Delta t$. Thus, we expect the total distance travelled by the object to be obtained by adding up the areas of many such time intervals.

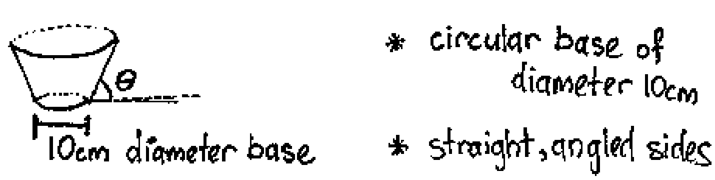
Making calculations of this sort rigourous leads to **integral calculus**. ##

We'll learn methods of calculus which let us solve the following problems:

1. Recover the acceleration and distance travelled from a velocity-time relationship.

2. Approximate $\sqrt{2}$ by a sequence of "easy to compute" fractions which converge very quickly.

3. Suppose that we wish to construct a flower-pot which holds exactly two litres of soil. If it has the proportions indicated in the diagram,



   what angle θ gives minimal surface area (and hence minimal materials cost)?

4. How do you minimize interest costs on a (student) loan?

Sketch of solution strategies:

Problem 2. Let $a = \sqrt{2}$ so $a^2 = 2$ and hence a is a *root* of the equation $f(x) = x^2 - 2$. This root-finding problem can be approached by "iterative guess-work" where the successive guesses are motivated by calculus, *i.e.,*
(i) use the slope of the graph at a guessed value of a_n [with $f(a_n) \neq 0$] to estimate where the *tangent line* cuts the x-axis.
(ii) use this point as the next guess a_{n+1}. ■

Problem 3. Derive a formula for how the volume of the flower pot is related to the angle θ and height h. Requiring this volume to be 2 litres will create a relationship between h and θ; this effectively makes h into a *function* of θ. The surface area (which would normally be expressed in terms of h and θ) could then be seen to depend only on θ, giving a function to be minimized. We'll learn enough calculus to be able to do this! ■

§ 0.2 Main Topics

A: Functions, Limits, and Continuity

B: Differentiation

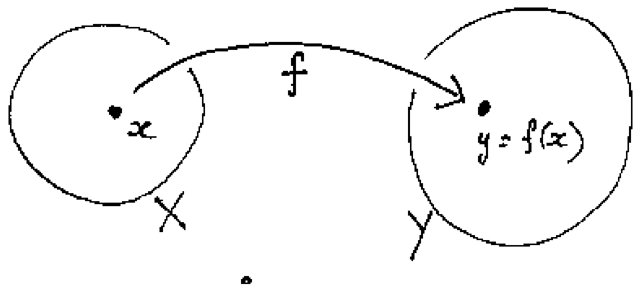
C: Integration

§1: Functions (cf Textbook §1: Prelim Concepts for Calculus)

A function is a rule for converting numbers from one set (the *domain*) into numbers in another set (the *range*).

$$x \mapsto f(x)$$

often write $y = f(x)$



Ex. 1

Suppose $f(x) = x^2 - 2$.

Try applying at

$$\begin{aligned} x = 1: &\quad y = f(1) &&= 1^2 - 2 \\ & &&= -1 \\ x = 2: &\quad y = f(2) &&= 2^2 - 2 \\ & &&= 4 - 2 \\ & &&= 2 \\ x = \sqrt{2}: &\quad y = f(\sqrt{2}) &&= (\sqrt{2})^2 - 2 \\ & &&= 0 \end{aligned}$$

Thus, f turns

1 into -1
2 into 2
$\sqrt{2}$ into 0
etc.

##

Precise defn:

Defn: A function $f : X \to Y$,
which is read as "f maps the set X into the set Y" (or for short f maps X to Y),
is a rule for assigning a *unique* element y from a set Y to each x from the set X. ■

Notation: We use the notation $x \in X$ to mean "x is a member of the set X" and write

$$y = f(x)$$

to mean "y is the (unique) member of the set Y which is obtained by applying f to x". We denote this relation by

$$f : x \mapsto y.$$

Defn: The set X is called the **domain** of f and Y is called the **range** of f. ■

Usually the domain is specified, but you may have to determine the range yourself.