Applied Mathematics

Course Outline

1

The aim of this course is to introduce several important methods for solving partial differential equations:

- Separation of variables
- Integral transforms (Laplace and Fourier)
- Complex variable methods
- Characteristics

1 Separation of variables

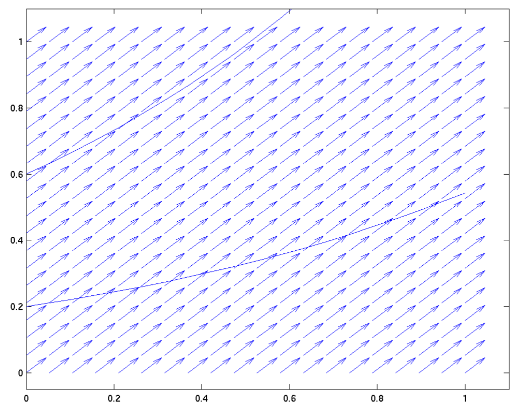
The first part of this section revises key ideas on ordinary differential equations and introduces the idea of a Green's function. Then we look at Sturm-Liouville theory and eigenfunction expansions, separation of variables, special functions including Bessel functions and Legendre functions.

1.1 Ordinary differential equations

The general form of a first-order equation for a function $y(x)$ is

$$y' = f(x, y) \tag{1.1}$$

where f is some function of the independent variable x and also of y. It is important to understand how such a differential equation determines the unknown function $y(x)$. The right-hand side of (1.1) is the slope of the graph of $y(x)$ versus x. At each point in the (x, y)-plane we can draw a small arrow representing this slope and these arrows define the *direction field* of the differential equation.



The diagram above shows the direction field for the exponential growth equation, $y' = y$. Here the slope of each arrow is just y. When $y = 0$ the arrows have slope zero and at $y = 1$ they are inclined at 45^o to the x-axis. If we start at some point in the plane and and move always in the direction of the arrows, the path we trace out will be the graph of a solution of the differential equation. For example beginning at the point $(0, a)$ on the y-axis we trace out the solution $y = a\exp(x)$. Two such solutions — for $a = 0.2$ and $a = 0.6$ — are shown in the diagram.

The solution depends on the starting point, but once this is specified the solution is *unique*. We conclude that the problem,

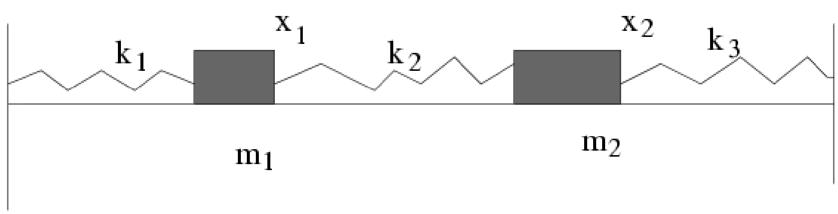
$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{1.2}$$

has a unique solution for the function $y(x)$. The graph of this solution is traced out by following the direction field, starting at the initial point (x_0, y_0). The problem (1.2) is called an *initial-value* problem.

Other problems may involve coupled systems of differential equations. For example, the diagram below shows a system of two masses m_1, m_2, moving on a frictionless horizontal surface and connected by springs with spring constants k_1, k_2, and k_3 The distances of the masses from the the origin $x = 0$ are given by x_1 and x_2.

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = k_2(x_2 - x_1) - k_1 x_1$$
$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = -k_3 x_2 - k_2(x_2 - x_1)$$

Here we have two unknown functions x_1, x_2, and two equations. The independent variable is time t.



A general coupled system can be written in the form,

$$\frac{\mathrm{d}\boldsymbol{Y}}{\mathrm{d}x} = \boldsymbol{F}(x, \boldsymbol{Y}). \tag{1.3}$$

Here,

$$\boldsymbol{Y} = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_n(x) \end{pmatrix}$$

is a vector of unknown functions and,

$$\boldsymbol{F} = \begin{pmatrix} f_1(x\,,\, y_1\,,\, y_2\,,\, \ldots y_n) \\ f_2(x\,,\, y_1\,,\, y_2\,,\, \ldots y_n) \\ \cdots \\ f_n(x\,,\, y_1\,,\, y_2\,,\, \ldots y_n) \end{pmatrix},$$

where the f_i are functions of the independent variable x and of the y_i. It is not so easy to draw direction fields in many dimensions but it can be shown that the system has a unique solution, given initial conditions for the unknown functions.

Theorem: The system (1.3) with given initial conditions

$$\boldsymbol{Y}(x_0) = \boldsymbol{Y}_0,$$

has a unique solution.

Remember that to deal with second-order equations we write them as a first order system. For example damped simple harmonic oscillator

$$\ddot{x} + k\dot{x} + x = 0 \tag{1.4}$$

(where the dot as usual denotes differentiation with respect to time) can be written as a system by setting,

$$x_1 = x \qquad x_2 = \dot{x}.$$

Then (1.4) is equivalent to the first-order system:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_2 - x_1.\end{aligned}$$

Linear second-order equations

Introductory courses on differential equations tend to focus on linear second-order equations since many can be solved quite easily. The general form of a second-order linear equation is,

$$y'' + a(x)y' + b(x)y = f(x), \tag{1.5}$$

where $a(x)$, $b(x)$ and $f(x)$ are given functions. If $f(x) = 0$ the equation is said to be *homogeneous*, in which case it can be shown that there are two independent solutions $y_1(x)$ and $y_2(x)$ say, and the general solution can be written as a linear combination of the two,

$$y(x) = Ay_1(x) + By_2(x), \qquad A\,,\ B \ \text{ constants.}$$

If $a(x)$ and $b(x)$ are constants the two independent solutions of the homogeneous equation can be easily found via the trial solution $y = \exp(\alpha x)$ where α is a constant. If $f(x) \neq 0$ then the trick is usually to guess a particular solution say $p(x)$; then the general solution of (1.5) is

$$y(x) = p(x) + Ay_1(x) + By_2(x),$$

the constants A and B being chosen so that $y(x)$ satisfies prescribed initial conditions.

If $y_1(x)$ and $y_2(x)$ are solutions of (1.5) their *Wronskian* is defined by setting

$$W(y_1\,,\,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{1.6}$$

Amazingly the Wronskian is always given by a simple formula which is (almost) independent of the solutions $y_1(x)\,,\,y_2(x)$. Differentiating (1.6) with respect to x we find,

$$W' = \begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}.$$

The first determinant vanishes because the rows are identical, and using (1.5) we can write the second as

$$\begin{vmatrix} y_1 & y_2 \\ -ay_1' - by_1 & -ay_2' - by_2 \end{vmatrix}.$$

Now adding $b\times$ (row 1) to (row 2) shows that

$$W' = -a(x)W.$$

and integrating,

$$\frac{W'}{W} = -a(x)$$

and integrating gives

$$\ln(W) - \int a(x) + C$$

where C is a constant. Now taking exponentials,

$$W = C\exp(-\int a(x)\mathrm{d}x). \tag{1.7}$$

where A is a constant. Actually this constant is the only part of formula (1.7) which depends on the solutions $y_1\,,\,y_2$. Most importantly, if $A = 0$ then $W = 0$ and the two solutions are not independent — one being simply a multiple of the other.

If we know one independent solution of (1.5) then (1.7) provides us with the second.

$$W = y_1y_2' - y_2y_1' = y_2^2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_1}{y_2}\right).$$

Thus if we know say $y_2(x)$ the other solution $y_1(x)$ is then given by the formula

$$y_1(x) = y_2(x)\int [y_2(x)]^{-2}\,W(x)\mathrm{d}x, \tag{1.8}$$

so long as we can evaluate the integral. (Even if we can't this formula often gives useful information about the second solution.)

As an example consider the equation

$$y'' + 2y' + y = 0.$$

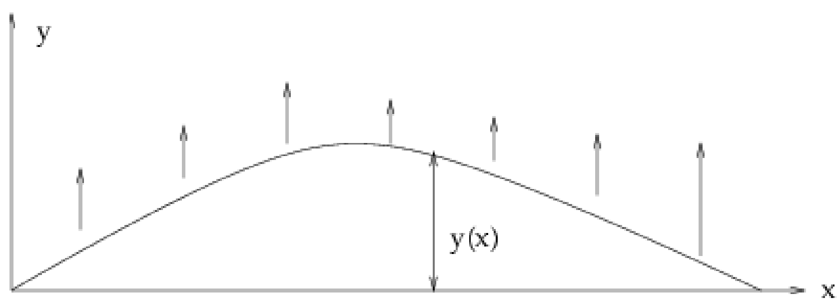
The usual trial solution $y = \exp(\alpha x)$ gives $(\alpha + 1)^2 = 0$. Because the quadratic is a perfect square we get only one solution, namely $\exp(-x)$. The coefficients in the equation are $a = 2$, $b = 1$ so (1.7) gives $W = Ae^{-2x}$. Now taking $y_2 = \exp(-x)$ in (1.8) we obtain,

$$y_1(x) = A\exp(-x)\int \exp(2x)\exp(-2x)\mathrm{d}x = Ax\exp(-x).$$

The second solution is therefore $x\exp(-x)$.

Green's functions

We begin by looking at a string stretched under tension along the x-axis between the points $x = 0$, $x = \ell$. We suppose that a force distribution acts on the string and distorts its shape, the force applied to a small element $[x\,,\, x + \mathrm{d}x]$ being $f(x)\mathrm{d}x\,\widehat{\boldsymbol{y}}$.



Consider a small length $\mathrm{d}s$ of the string; suppose that $\gamma(s)$ is the tension and $\widehat{\boldsymbol{T}}(s)$ the unit tangent vector.

The force per unit length acting on the string is

$$\frac{\gamma(s + \mathrm{d}s)\widehat{\boldsymbol{T}}(s + \mathrm{d}s) - \gamma(s)\widehat{\boldsymbol{T}}(s)}{\mathrm{d}s}.$$

Letting $\mathrm{d}s \to 0$ we see that the force per unit length is

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\gamma(s)\widehat{\boldsymbol{T}}(s)\big] = \frac{\mathrm{d}\gamma}{\mathrm{d}s}\widehat{\boldsymbol{T}} + \gamma\kappa\widehat{\boldsymbol{N}},$$

where κ is the curvature of the string and $\widehat{\boldsymbol{N}}$ the principal normal. We now make the simplifying assumption that the slope of the string is small, so that $y'(x) \ll 1$. This means that we can take $\widehat{\boldsymbol{T}} = \widehat{\boldsymbol{x}}$, $\widehat{\boldsymbol{N}} = \widehat{\boldsymbol{y}}$, while the curvature

$$\kappa = \frac{y''}{[1+(y')^2]^{3/2}} \approx y''.$$

The tension force and the applied force must balance, so

$$f(x)\,\widehat{\boldsymbol{y}} + \frac{\mathrm{d}\gamma}{\mathrm{d}s}\,\widehat{\boldsymbol{x}} + \gamma\, y''\,\widehat{\boldsymbol{y}} = 0.$$

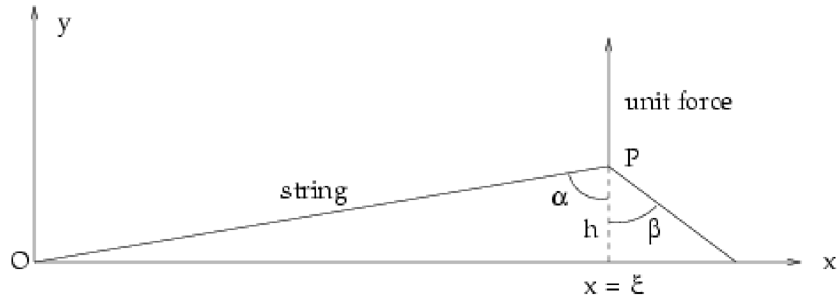
We conclude that the tension γ is constant and for convenience set $\gamma = 1$. Then the x-component of the above equation then gives,

$$y'' = -f(x). \tag{1.9}$$

The displacement $y(x)$ must also satisfy two boundary conditions,

$$y(0) = y(\ell) = 0. \tag{1.10}$$

Equations (1.9) and (1.10) are quite easy to solve by direct integration, but we want to approach this problem from another angle.



The above diagram shows the effect of a unit force acting at the point P, $x = \xi$ of the string. The forces acting at P must balance, so

$$\cos\alpha + \cos\beta = 1.$$

Now $\cos\alpha = h/OP \approx h/\xi$ since the slope of the string is small. Similarly we can show that $\cos\beta \approx h/(\ell - \xi)$ so

$$\frac{h}{\xi} + \frac{h}{\ell - \xi} = 1, \qquad h = \frac{\xi(\ell - \xi)}{\ell}.$$

This displacement of the string due to a unit point force applied at $x = \xi$ is written as $G(x, \xi)$. We call G the *Green's function* for the problem. Specifically,

$$G(x,\xi) = \begin{cases} \dfrac{hx}{\xi} = \dfrac{(\ell - \xi)x}{\ell} & \text{if } 0 \le x \le \xi; \\ \\ \dfrac{h(\ell - x)}{\ell - \xi} = \dfrac{\xi}{\ell}(\ell - x) & \text{if } \xi \le x \le \ell. \end{cases} \tag{1.11}$$

Now we subdivide the string into n equal intervals $[\xi_i\,,\,\xi_{i+1}]$ of width $\Delta\xi$, where

$$\xi_0 = 0, \qquad \xi_n = \ell, \qquad \Delta\xi = \ell/n.$$

The force on the interval $[\xi_i\,,\,\xi_{i+1}]$ is $f(\xi_i)\Delta\xi$. Assuming $\Delta\xi$ is small we can consider this interval as a point and the force as a *point force*. The string displacement due to the force on the small interval is therefore $G(x, \xi_i)f(\xi_i)\Delta\xi$. Summing up the effects of the forces acting on each interval we obtain the total displacement,

$$y(x) = \sum_{i=1}^{n} G(x, \xi_i) f(\xi_i) \Delta\xi.$$

In the limit as $\Delta\xi \to 0$ this sum becomes the integral,

$$y(x) = \int_0^{\ell} G(x, \xi) f(\xi) \mathrm{d}\xi.$$

The big advantage of this formula is that it provides the solution for *any* force distribution $f(x)$. For example suppose that the applied force is simply the weight of the string, so that $f(x) = -g\lambda$ where λ is the string mass per unit. The displacement

$$y(x) = -g\lambda \int_0^\ell G(x, \xi)\mathrm{d}\xi.$$

Since the Green's function is given by different formula for different ranges of ξ-values we split the integral into two pieces,

$$\int_0^\ell G(x, \xi)\mathrm{d}\xi = \int_0^x G(x, \xi)\mathrm{d}\xi + \int_x^\ell G(x, \xi)\mathrm{d}\xi.$$

In the first integral, $\xi \leq x$ so we use the *second* of the formulae (1.11). Similarly for the second integral we use the first formula, so

$$y(x) = -g\lambda \int_0^x \frac{\xi}{\ell}(\ell - x)\mathrm{d}\xi - g\lambda \int_x^\ell \frac{(\ell - \xi)x}{\ell}\mathrm{d}\xi.$$

Since

$$\int_0^x \xi \mathrm{d}\xi = \tfrac{1}{2}x^2 \qquad \text{and} \qquad \int_x^\ell (\ell - \xi)\mathrm{d}\xi = \left[-\tfrac{1}{2}(\ell - \xi)^2\right]_x^\ell = \tfrac{1}{2}(\ell - x)^2,$$

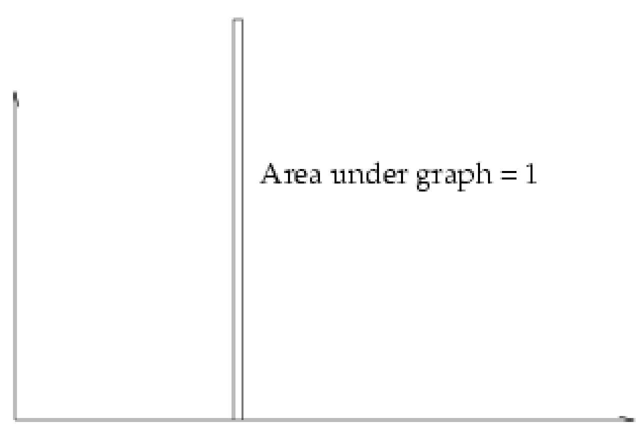
we find that

$$y(x) = -\tfrac{1}{2}g\lambda x(\ell - x).$$

The string sags under its own weight in an inverted parabola.

The Dirac delta function

Our Green's function $G(x, \xi)$ described the effect of a unit point load on the string. We can think of a unit point load as a distributed load applied over a very narrow interval. The integral of the load over that narrow interval must equal the total force, which is unity. The graph of this force distribution would therefore look something like:



Such a function is called a “delta function”, $\delta(x)$, defined the properties,

$$\delta(x) = 0 \text{ if } x \neq 0; \qquad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1.$$

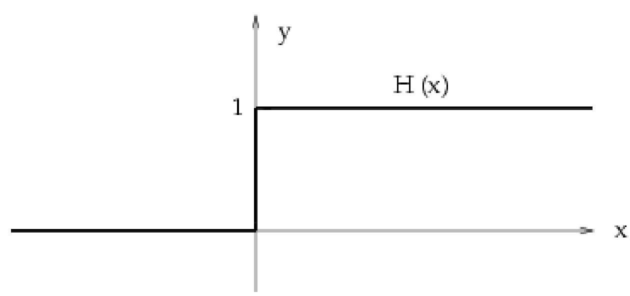
Actually this definition is self-contradictory, but essentially we picture a delta function as having a very large value over a very small interval in such a way that the area under its graph = 1. (To define such a function precisely we need to broaden the usual definition of a function to include “generalised functions’ — but that is another story.) An important property of the delta function is that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = \int_{-\infty}^{\infty} f(a)\delta(x-a)\mathrm{d}x = f(a)$$

for any function $f(x)$. (If $x \neq a$ the delta function is zero so there is no contribution to the integral.)

The delta function is closely related to the *Heaviside* function $H(x)$ defined by setting,

$$H(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$



Now $H'(x) = 0$ if $x \neq 0$ and

$$\int_{-\infty}^{\infty} H'(x)\mathrm{d}x = H(\infty) - H(-\infty) = 1,$$

so $H'(x)$ has precisely the same defining properties as $\delta(x)$.

$$H'(x) = \delta(x). \tag{1.12}$$

A unit load on the string applied at the point $x = \xi$ corresponds to a load distribution function $f(x) = \delta(x - \xi)$. The string displacement due to this point load is $G(x, \xi)$ so we can think of $G(x, \xi)$ as the solution of the problem,

$$y'' = -\delta(x - \xi).$$

For convenience we write a general linear second-order equation in the form,

$$L(y) = f(x), \qquad \text{where} \qquad L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a(x)\frac{\mathrm{d}}{\mathrm{d}x} + b(x)$$

is a linear differential operator.

Definition:
The Green's function $G(x, \xi)$ for the operator L over the interval $[a\,,\,b]$ with boundary conditions

$$B_1(y, a) = B_2(y, b) = 0,$$

is the solution of the problem,

$$L(G) = \delta(x - \xi), \qquad B_1(G, a) = B_2(G, b) = 0. \tag{1.13}$$

Here B_1 and and B_2 are *homogeneous* boundary conditions of the form,

$$c_1 y(a) + c_2 y'(a) = 0, \qquad c_3 y(b) + c_4 y'(b) = 0.$$

where the c_i are constants.

Theorem:
The solution of the boundary-value problem,

$$L(y) = f(x), \qquad B_1(y, a) = B_2(y, b) = 0$$

is

$$y(x) = \int_a^b G(x, \xi) f(\xi) \mathrm{d}\xi. \tag{1.14}$$

To prove this theorem we first note that $B_1(y, a) = B_2(y, b) = 0$ since G satisfies the same boundary conditions. Also

$$L(y) = \int_a^b L(G)\, f(\xi)\, \mathrm{d}\xi = \int_a^b \delta(x - \xi)\, f(\xi)\, \mathrm{d}\,\xi = \; f(x).$$

The problem now of course is how do we solve the mysterious equation (1.13) to produce the Green's function. Looking back at our Green's function for the stretched string we see that $G(x, \xi)$ was defined by different formulae in the intervals $[0\,,\,\xi\,]$ and $[\xi\,,\,\ell\,]$.The graph of $G(x, \xi)$ at the top of page 8 shows that the Green's function is continuous in x but the first derivative has a discontinuity at $x = \xi$. In fact we can write,

$$G\,' = \frac{\ell - \xi}{\ell} - H(x - \xi),$$

so that

$$L(G) = G'' = H'(x - \xi) = \delta(x - \xi).$$

Thus the delta function is produced by differentiating the unit discontinuity in G'.

Now we can formulate a recipe for finding the Greens function. It is given by separate formulae, $G_1(x, \xi)$ say in the interval $[a\,,\,\xi]$ and $G_2(x, \xi)$ in $[\xi\,,\,b]$. It must have the following properties:

$$L(G_1) = L(G_2) = 0; \qquad B_1(G_1, a) = 0; \qquad B_2(G_2, b) = 0. \tag{1.15}$$

$$G_1(x, \xi) = G_2(x, \xi) \qquad \text{(continuity)}. \tag{1.16}$$

$$G_2'(x, \xi) - G_1'(x, \xi) = 1 \qquad \text{(unit jump in the first derivative)}. \tag{1.17}$$

EXAMPLE 1: The end $x = 0$ of a uniform metal bar is thermally insulated and the end $x = a$ is maintained at a constant temperature 0. The temperature $u(x)$ in the bar therefore satisfies

$$u'' = -f(x), \qquad u'(0) = 0, \quad u(a) = 0 \tag{1.18}$$

where $f(x)$ is a heat source distribution. Find $u(x)$ using a Green's function method.

We take the Green's function as $G_1(x, \xi)$ in the interval $[0\,,\,\xi]$ and $G_2(x, \xi)$ in $[\xi\,,\,a]$. Thus G_1 must satisfy,

$$G_1'' = 0, \qquad G_1'(0, \xi) = 0.$$

The differential equation shows that $G_1 = A + Bx$ where A and B are constants, and the condition at $x = 0$ gives $B = 0$.

$$G_1 = A.$$

Similarly

$$G_2'' = 0, \qquad G_2'(a, \xi) = 0.$$

We have $G_2 = D + Cx$ where D and C are constants, and the condition at $x = a$ shows that

$$G_2 = C(x - a).$$

Applying the continuity condition (1.16) and the jump condition (1.17) give,

$$A = C(\xi - a), \qquad C = 1.$$

Solving for the constants A and C we find,

$$G_1(x,\xi) = \xi - a, \qquad G_2(x,\xi) = x - a.$$

The solution of the boundary-value problem (1.18) is given by

$$y(x) = -\int_0^a f(\xi)G(x,\xi)\mathrm{d}\xi = -\int_0^x f(\xi)(x-a)\mathrm{d}\xi - \int_x^a f(\xi)(\xi - a)\mathrm{d}\xi.$$

This solution works for any heat source function $f(x)$. In particular suppose $f(x) = f_0$, a constant. Then we find

$$y(x) = -f_0 x(x-a) - f_0\left[\tfrac{1}{2}(a^2 - x^2) - a(a-x)\right] = \tfrac{1}{2}f_0(a^2 - x^2).$$

It is easily verified by substitution into (1.18) that the solution is correct.

The Green's function method can also be used on semi-infinite intervals, with both boundary conditions specified at one end.

EXAMPLE 2: Solve:

$$\ddot{x} + \omega^2 x = f(t), \qquad x(0) = \dot{x}(0) = 0. \tag{1.19}$$

using a Green's function method.
Here the interval is $0 \le t < \infty$ with boundary conditions at $t = 0$. We let $G_1(t,\xi)$ be the Green's function in the interval $0 \le t \le \xi$ and $G_2(t,\xi)$ in the interval $t > \xi$, so G_1 must satisfy,

$$\ddot{G}_1 + \omega^2 G_1 = 0, \qquad G_1(0,\xi) = \dot{G}_1(0,\xi) = 0.$$

The solution of this problem is simply $G_1 = 0$. As for G_2,

$$\ddot{G}_2 + \omega^2 G_2 = 0,$$

and there are no boundary conditions to apply, so the general solution is

$$G_2 = A\sin(\omega t) + B\cos(\omega t).$$

The continuity condition (1.16) and the jump condition (1.17) give,

$$\text{(i)}: A\sin(\omega\xi) + B\cos(\omega\xi) = 0, \qquad \text{(ii)}: \omega A\sin(\omega\xi) - \omega B\sin(\omega\xi) = 1.$$

Multiplying (i) by $\sin\omega\xi$, (ii) by $\cos(\omega\xi)/\omega$, and adding gives, $A = \cos(\omega\xi)/\omega$. Then substituting into (i) gives $B = -\sin(\omega\xi)/\omega$. Thus,

$$G_2(t,\xi) = \frac{1}{\omega}\cos(\omega\xi)\sin(\omega t) - \frac{1}{\omega}\sin(\omega\xi)\cos(\omega t) = \frac{1}{\omega}\sin\big[\omega(t-\xi)\big].$$

The solution of (1.19) is

$$x(t) = \frac{1}{\omega}\int_0^t \sin\big[\omega(t-\xi)\big] f(\xi)\mathrm{d}\xi.$$

For example take $f(t) = \epsilon\sin\omega t$ where ϵ is a constant (resonance). Then

$$\begin{aligned}
x(t) &= \frac{\epsilon}{\omega}\int_0^t \sin\big[\omega(t-\xi)\big]\sin\omega t\mathrm{d}\xi \\
&= \frac{\epsilon}{2\omega}\int_0^t \big[\cos(\omega t - 2\omega\xi) - \cos(\omega t)\big]\mathrm{d}\xi \\
&= \frac{\epsilon}{2\omega}\left[-\frac{1}{2\omega}\sin(\omega t - 2\omega\xi) - \xi\cos\omega t\right]_0^t \\
&= \frac{\epsilon}{2\omega}\left[\frac{1}{\omega}\sin\omega t - t\cos\omega t\right].
\end{aligned}$$

Note the linear growth of the solution with time.

Green's function for Poisson's equation

Green's functions can also be used to solve partial differential equations, for example Poisson's equation,

$$\nabla^2\phi(\boldsymbol{x}) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = f(\boldsymbol{x})$$

where $\boldsymbol{x}$ is shorthand notation for (x, y, z). This is one one of Maxwell's equations; $\phi(\boldsymbol{x})$ is the electric potential and $f(\boldsymbol{x})$ the electric charge distribution (apart from a few dimensional factors). The Green's function $G(\boldsymbol{x}, \boldsymbol{\xi})$ for the Laplacian operator ∇^2 is the solution of

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{\xi}).$$

Here $\delta(\boldsymbol{x})$ is a 3D delta function with the properties:

$$\delta(\boldsymbol{x}) = 0 \;\text{ if }\; \boldsymbol{x} \neq 0, \qquad \int \delta(\boldsymbol{x})\mathrm{d}V = 1,$$

where the integration is taken over all space. It can be shown (see problem sheet 2) that the Green's function for the Laplacian operator is given by

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{\xi}|}.$$

Thus the solution of Poisson's equation is

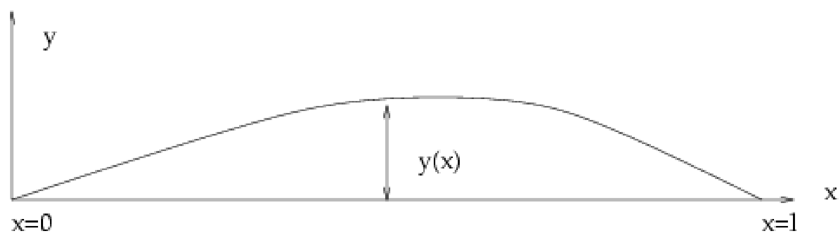
$$\phi(\boldsymbol{x}) = -\frac{1}{4\pi} \int_V \frac{f(\boldsymbol{\xi}) \mathrm{d}V_\xi}{|\boldsymbol{x} - \boldsymbol{\xi}|}.$$

The subscript ξ on $\mathrm{d}V_\xi$ indicates that the integration is with respect to the $\boldsymbol{\xi}$ variable.

1.2 Sturm Liouville Problems

Note: This subject is covered in Boyce and DiPrima, *Elementary differential equations and boundary-value problems* Chapter 11.

Review of vibrating string problem



We begin this section with a review of the analysis of a vibrating string. We found in the last section that if we assume the displacement of the string is small (compared to its length) then the tension force per unit length is $\gamma y''$ where γ is the tension. By Newton's second law, this must equal the mass acceleration per unit length of string, so

$$\gamma y_{xx} = \rho y_{tt}$$

where ρ is the mass per unit length of string. This can be written in the standard wave equation form,

$$y_{tt} = c^2 y_{xx}, \qquad c = \sqrt{\gamma/\rho}. \tag{1.20}$$

The constant c is the speed of propagation of waves along the string.

Suppose the string has an initial deformation $y(x,0) = f(x)$ say, and is released from rest. To determine the subsequent motion of the string we must find a solution of (1.20) which satisfies the above initial condition, and also the end conditions $y(0) = y(\ell) = 0$. We use the method of *separation of variables* — i.e. we first look for solutions of the form,

$$y = X(x)T(t),$$

where X is a function of x only and T a function of time t only. Substituting this trial solution into (1.20) gives,

$$\ddot{T}X = c^2 X''T.$$

Dividing each side by XT,

$$\frac{\ddot{T}}{c^2T} = \frac{X''}{X} = \quad \text{a constant} \quad = -\lambda^2 \;\; \text{say},$$

since the left-hand side is a function of t only and the right-hand side a function of x only. Solving for X we have

$$X'' + \lambda^2 X = 0, \qquad X(0) = X(\ell) = 0. \tag{1.21}$$

This is our first example of a Sturm-Liouville problem. Solutions exist only if $\lambda = n\pi/\ell$, where n is an integer, and they take the form,

$$X = \sin\left(\frac{n\pi x}{\ell}\right).$$

These solutions are called the *eigenfunctions* of the *eigenvalue problem* (1.21). The corresponding solutions for T which satisfy the zero initial condition are

$$T(t) = \cos\left(\frac{n\pi ct}{\ell}\right),$$

In general the solution will be a sum of products of the elementary separated solutions,

$$y = \sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right).$$

The constants a_n are chosen using the theory of Fourier series so as to satisfy the initial condition, $y(x,0) = f(x)$ – i.e.

$$\sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x). \tag{1.22}$$

The key to finding the coefficients is that the eigenfunctions

$$\sin\left(\frac{n\pi x}{\ell}\right) = E_n(x) \;\; \text{say},$$

are *orthogonal.* In other words

$$\int_0^\ell E_n(x)E_m(x)\mathrm{d}x = 0 \quad \text{if} \quad m \neq n.$$

More precisely and concisely we can write

$$\langle E_n\,,\,E_m\rangle = \tfrac{1}{2}\ell\delta_{nm}$$

where the symbol δ_{mn} (or Kronecker delta) is defined by

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

and the angle bracket notation means an integral over the interval $[0\,,\,\ell]$,

$$\langle f\,,\,g\rangle = \int_0^\ell f(x)g(x)\mathrm{d}x.$$

This integral can be thought of as a generalised dot product — the values of the functions at all points of the interval are multiplied together and summed, like the components of two vectors in a dot product.

The orthogonality makes it easy to calculate the coefficients a_n in (1.22). Rewriting this equation as,

$$\sum_{n=1}^{\infty} a_n E_n = f(x)$$

and taking the inner product of each side with E_k we find that all the inner products on the LHS vanish except when $n = k$. Thus

$$\tfrac{1}{2}\ell a_k = \langle E_k\,,\,f\rangle, \qquad a_k = \frac{2}{\ell}\langle E_k\,,\,f\rangle.$$

1.3 self-adjoint operators

The method of separation of variables always gives rise to Sturm-Liouville problems of the form (1.21). In this section we need to consider the more general version,

$$L(y) = 0, \qquad a \leq x \leq b, \qquad B_1(y,a) = B_2(y,b) = 0, \tag{1.23}$$

where L is the differential operator,

$$L(y) = a_2(x)y'' + a_1(x)y' + a_0(x)y.$$

The boundary conditions B_1, B_2 are as defined on page 10 of §1.2.

It is usually simpler to write the operator L is *self-adjoint* form: This

$$(p(x)y')' + q(x)y = 0. \tag{1.24}$$

This can be accomplished as follows. Consider the general second-order linear operator of the form,

$$L(u) = A_1u'' + A_2u' + A_3u. \tag{1.25}$$

Note that A_1, A_2 and A_3 are in general functions of x. To express this in self-adjoint form we multiply each side by

$$\mu(x) = \frac{p(x)}{A_1(x)}$$

to give

$$p(x)u'' + \mu A_2u' + \mu A_3u.$$

We want

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)u'\right] = pu'' + \mu A_2u',$$

i.e.

$$\frac{\mathrm{d}p}{\mathrm{d}x}u' + pu'' = pu'' + \mu A_2u'$$

which simplifies to

$$\frac{\mathrm{d}p}{\mathrm{d}x}u' = \mu A_2u' \quad \text{or} \quad \frac{\mathrm{d}p}{\mathrm{d}x} = \mu A_2 = \frac{p(x)A_2(x)}{A_1(x)}.$$

Thus

$$\frac{\mathrm{d}p}{p} = \frac{A_2(x)}{A_1(x)}, \quad p(x) = c\exp\left(\int \frac{A_2}{A_1}\mathrm{d}x.\right). \tag{1.26}$$

EXAMPLE 1: Write the operator

$$L(y) = x^2y'' + xy' + y$$

In self-adjoint form. Since $A_2/A_1 = 1/x$ (1.26) gives $p(x) = x$ and $\mu = 1/x$.Multiplying each side by μ gives

$$xu'' + u' + (u/x) = 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[xu'\right] + u/x = 0$$

in self adjoint form.

We now prove a useful identity for self adjoint operators.

Theorem: (Green's identity). For a self-adjoint operator L of the form (1.24),

$$, \int_a^b \big[uL(v) - vL(u)\big]\mathrm{d}x = \big[p(uv' - vu'\big]_a^b. \tag{1.27}$$

where $u(x)$ and $v(x)$ are any functions.

Proof:

$$\begin{aligned} &uL(v) - vL(u) = u(pv')' - v(pu')' \\ &= (puv)' - pu'v' - (puv)' + pu'v' \\ &= (puv')' - (pvu')' \\ &= \big[p(uv' - vu')\big]'. \end{aligned}$$

The result follows by integration.

Eigenvalue problems

When we look for separated solution of a partial differential equation the separation constant λ will appear in the resulting ordinary differential equation. For example, in the vibrating string problem we had solve the problem,

$$y'' + \lambda^2 y = 0, \qquad y(0) = y(\ell) = 0.$$

We then found that a solution of this boundary-value problem was possible only certain values of λ, namely the *eigenvalues* $\lambda = n\pi/\ell$. The corresponding solutions, $\sin(n\pi x/\ell)$ are called *eigenfunctions*.

More generally we'll consider eigenvalue problems in the self-adjoint form,

$$[p(x)y']' + [r(x) + \lambda w(x)]y = 0. \tag{1.28}$$

EXAMPLE: Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem,

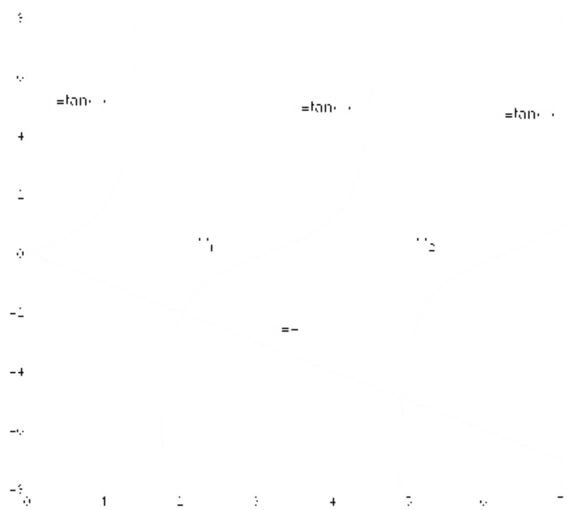
$$y'' + y = 0; \qquad y(0) = 0, \qquad y(1) + y'(1) = 0.$$

For convenience we write $\lambda = \omega^2$. Then

$$y = A\sin(\omega x) + B\cos(\omega x)$$

where A and B are constants. The first boundary condition implies that $y(0) = B = 0$, and the second that, $A\sin\omega + \omega A\cos\omega = 0$ or

$$\omega + \tan(\omega) = 0. \tag{1.29}$$



The figure on the following page shows graphically that the transcendental equation (1.29) has an infinite number of solutions for ω. The first two are $\omega_1 = 2.028$ and $\omega_2 = 4.913$ approximately. For large n the distance between two successive solutions $\omega_{n+1} - \omega_n \to \pi$, the distance between the branches of $y = \tan(x)$.

Why do eigenvalue problems of the form (1.23) produce a sequence of eigenfunctions? Look again at the very simple problem

$$y'' + \lambda^2 y = 0, \qquad y(0) = y(\ell) = 0,$$

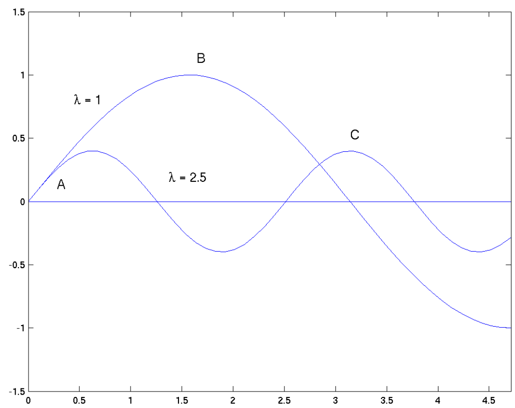
which can be written in the form,

$$y'' = -\lambda^2 y, \tag{1.30}$$

We can interpreted y'' as the curvature of the graph of the solution. More precisely the curvature

$$\kappa = \frac{y''}{(1 + (y')^2)^{3/2}},$$

but if y' is small $\kappa \approx y''$. In any case the signs of κ and y'' are always equal. Suppose we take $y' > 0$ and start drawing the graph of $y(x)$ with $\lambda = 1$, beginning at the origin. Because $y' > 0$ $y(x)$ will at first increase (point A); but it can be seen from (1.30) that $y'' < 0$ — i.e. the curvature of the graph is negative, so the slope will decrease and eventually fall to to zero (point B). The curvature remains negative so long as $y > 0$ so the slope then becomes negative and eventually the graph crosses the x-axis (point C) at which point the curvature vanishes. After this point $y(x)$ becomes negative and (1.30) now shows that the curvature will be positive. The so the graph begins to curve upwards again, and eventually re-crosses the x-axis — and this pattern is repeated.



The second graph shows the effect of increasing λ to 2.5, but keeping the same initial slope at $x = 0$. The curvature for a given y value is now $6.25\times$ the value for the previous graph, so this graph oscillates more rapidly.

Thus the parameter λ controls the rate of oscillation of the graph, and by choosing the correct value we can make the first zero of $y(x)$ occur at any given point. The first eigenvalue λ_1 for the string problem is chosen so that the first zero occurs at $x = \ell$; then the second eigenvalue is obtained by increasing λ until the *second* zero of $y(x)$ occurs at $x = \ell$, and so on.

Similar arguments can be used for the more general form of the eigenvalue problem, (1.28). The form of the equation can be simplified by transforming the independent variable x so that the operator

$$p(x)\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \mathrm{d}\xi = \frac{\mathrm{d}x}{p(x)}.$$

Integrating we find

$$\xi = \int \frac{\mathrm{d}x}{p(x)}.$$

Equation (1.28) can now be written in the form,

$$\frac{1}{p(x)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right) + \big[r(x) + \lambda w(x)\big]y = 0,$$

or,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \big[R(\xi) + \lambda W(\xi)\big]y = 0, \tag{1.31}$$

where $R(\xi) = p(x)(r(x),\ W(\xi) = p(x)w(x)$. For example, to transform the equation,

$$(xy')' + \frac{\lambda y}{x} = 0,$$

we set

$$\xi = \int \frac{\mathrm{d}x}{x} = \ln(x).$$

Then,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \lambda y = 0.$$

Oscillation theorem

Our argument for the existence of a sequence of eigenvalues and corresponding eigenfunctions for (1.31) is based on the fact that as the (positive) RHS term becomes larger, the solution oscillates more rapidly — i.e. the zeros of the solution move towards the origin. We can make the argument more precise by the following theorem.

Theorem: Suppose $y_1(x)$ and $y_2(x)$ satisfy the differential equations,

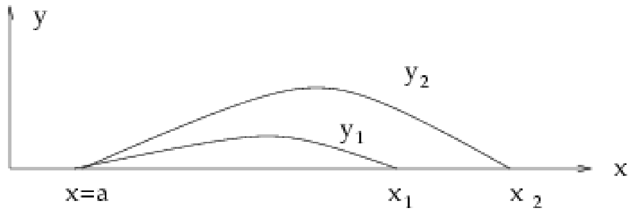
$$y_1'' + Q_1(x)y_1 = 0. \tag{1.32}$$

$$y_2'' + Q_2(x)y_2 = 0. \tag{1.33}$$

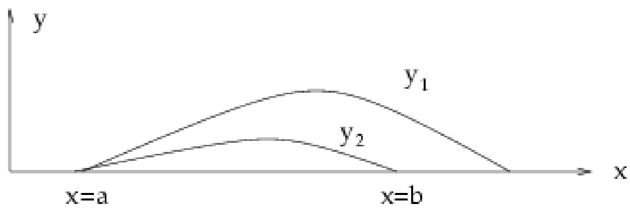
$$y_1(a) = y_2(a) = 0. \tag{1.34}$$

$$Q_1(x) > Q_2(x) > 0, \qquad \text{if} \ \ x \geq a. \tag{1.35}$$

Then if x_i is the first zero of y_i to the right of $x = a$, $x_1 < x_2$.



Proof: We use a proof by contradiction — i.e. suppose the y_2 has the first zero at $x = b$ say. We can assume without loss of generality that $y_1'(a) = y_2'(a) = 1$, since if for example we chose instead $y_1'(a) = \alpha$ this would simply multiply the solution by a factor α without changing the positions of any zeros.



Multiply (1.32) by y_2, (1.33) by y_1 and subtract.

$$y_2y_1'' - y_1y_2'' + (Q_1 - Q_2)y_1y_2 = 0.$$

This can be re-written,

$$\left(y_2y_1' - y_1y_2'\right)' + (Q_1 - Q_2)y_1y_2 = 0$$

and integrated from $x = a$ to $x = b$,

$$\left[y_2y_1' - y_1y_2'\right]_a^b + \int_a^b (Q_1 - Q_2)y_1y_2 \mathrm{d}x.$$

Using (1.34) and the fact that $y_2(b) = 0$ we find,

$$-y_1(b)y_2'(b) + \int_a^b (Q_1 - Q_2)y_1y_2 \mathrm{d}x = 0.$$

But both terms in this equation are positive, which is a contradiction. It must therefore be true that $x_1 < x_2$.

Corollary: If $Q(x) > M$ for all x, where M is a positive constant, the distance between consecutive zeros of the equation,

$$y'' + Q(x)y = 0,$$

is less than $\pi/\sqrt{M}$. To prove this we apply the last theorem to the two equations,

$$y'' + My = 0, \qquad y'' + Q(x)y = 0.$$

We conclude that the distance between consecutive zeros of the second equation is less than that for the first equation. But the general solution of the first equation is

$$y = A\cos(\sqrt{M}\,x + \alpha)$$

where A and α are constants, and the distance between its zeros is $\pi/\sqrt{M}$ — hence the result.

We can show similarly that if $Q(x) < M$ then the distance between consecutive zeros is greater than $\pi/\sqrt{M}$.

EXAMPLE: Find upper and lower bounds on the position of the first positive zero of $y(x)$ where,

$$[(1+x)y']' + y = 0; \qquad y(0) = 0.$$

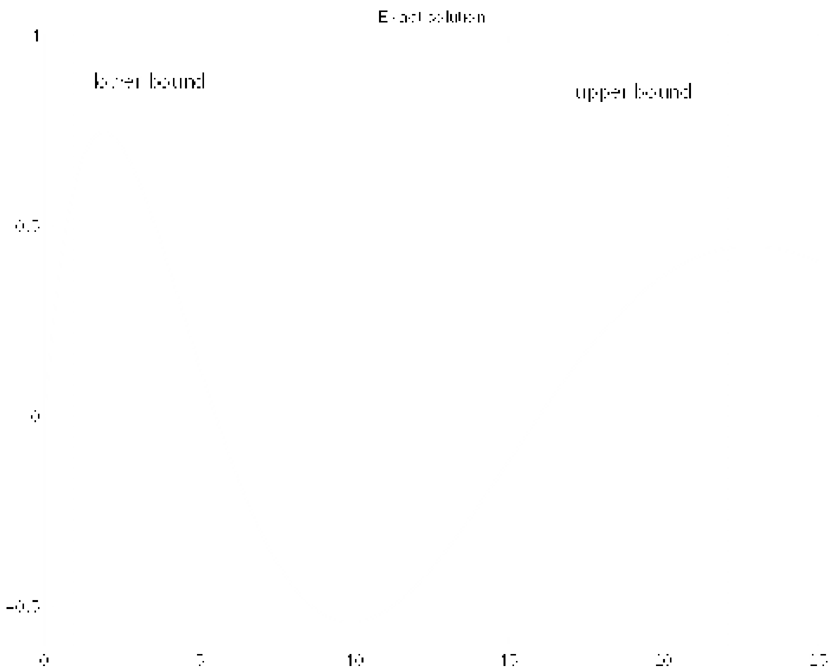
We first have to write this equation in our standard form. The transformed variable,

$$\xi = \int \frac{1}{p(x)}\mathrm{d}x = \int \frac{1}{1+x}\mathrm{d}x = \ln(1+x).$$

Thus $p(x) = 1 + x = e^{\xi}$ so the transformed equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} + e^{\xi}y = 0.$$

We can take $M = 1$ so the first zero occurs before $\xi = \pi$ or $x = e^{\pi} - 1$. In the interval $[0\,,\,\pi]$, $e^{\xi} < e^{\pi}$. We can then argue that the first zero occurs after $\xi = \pi/\sqrt{e^{\pi}}$ or $x = 0.921$.



As can be seen from the exact solution, these bounds are correct, but not very informative, because $Q(x)$ varies quite rapidly.

Orthogonality of eigenfunctions

Theorem 1: The Sturm-Liouville problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [R(x) + \lambda w(x)]y = 0, \qquad y(a) = y(b) = 0,$$

where the function $w(x)$ is positive, has an infinite sequence of eigenfunctions.

Proof: As λ increases $y(x)$ oscillates more rapidly. The n'th eigenvalue λ_n is such that the n'th zero of $y(x)$ coincides with $x = b$. The corresponding solution is the the n'th eigenfunction $E_n(x)$.

Note: This theorem holds for the more general boundary conditions, $B_1(y, a) = B_2(y, b) = 0$, but the proof is omitted.

Theorem 2: The sequence of eigenfunctions of the eigenvalue problem,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [R(x) + \lambda w(x)]y = 0, \tag{1.36}$$

$$B_1(y,a) = B_2(y,b) = 0, \tag{1.37}$$

is orthogonal with weight function $w(x)$ — i.e.

$$\int_a^b w(x) E_m(x) E_n(x) \mathrm{d}x = 0, \qquad m \neq n.$$

Proof: Let L be the self-adjoint operator,

$$L(y) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + R(x) y.$$

Then (1.36) for the m'th eigenfunction $E_m(x)$ takes the form.

$$L(E_m) = -\lambda_m w(x) y(x). \tag{1.38}$$

Similarly for the n'th eigenfunction,

$$L(E_n) = -\lambda_n w(x) y(x). \tag{1.39}$$

Multiplying (1.38) by E_n and (1.39) by E_m and subtracting we find that

$$E_n L(E_m) - E_m L(E_n) = (\lambda_n - \lambda_m) w(x) E_m(x) E_n(x).$$

Integrating over $[a\,,\,b]$ and using Green's identity, we find that

$$(\lambda_m - \lambda_n) \int_a^b w(x) E_m(x) E_n(x) \mathrm{d}x = \big[E_n E_m' - E_m E_n'\big]_a^b. \tag{1.40}$$

The RHS can be written as,

$$\left| \begin{array}{cc} E_n & E_m \\ E_n' & E_m' \end{array} \right|_a^b.$$

The boundary condition (1.37) at $x = a$ gives,

$$c_1 E_n(a) + c_2 E_n'(a) = 0, \qquad c_1 E_m(a) + c_2 E_m'(a) = 0.$$

Thus adding $c_1\times$(row 1) to $c_2\times$(row 2) of the determinant shows that it vanishes at $x = a$. Similarly it vanishes at $x = b$. Since $(\lambda_m - \lambda_n) \neq 0$ it follows from (1.40) that

$$\int_a^b w(x) E_m(x) E_n(x) \mathrm{d}x = 0.$$

The result of this theorem can be written simply as,

$$\langle E_m\,,\,E_n \rangle = \delta_{mn} \|E_n\|^2,$$

Here the inner product includes the positive weight function $w(x)$ so that

$$\langle E_m\,,\,E_n \rangle = \int_a^b w(x) E_n(x) E_m(x) \mathrm{d}x, \qquad \|E_n\|^2 = \langle E_n\,,\,E_n \rangle.$$

We now see that the sequences $\sin(n\pi x/\ell)$ or $\cos(n\pi x/\ell)$ are not the only kinds of orthogonal sequences. We've seen that any eigenvalue problem gives rise to an orthogonal sequence of eigenfunctions. These sequences can be shown to be *complete* — i.e. any function $f(x)$ can be expanded in terms of the $E_n(x)$ in the form,

$$f(x) = \sum_{n=1}^{\infty} a_n E_n(x),$$

where the a_n are suitably chosen coefficients. To find a_k say take the inner product of of the expansion with $E_k(x)$.

$$\begin{aligned}\langle f\,,\, E_k\rangle &= \sum_{n=1}^{\infty} a_n \langle E_n\,,\, E_k\rangle \\ &= \sum_{n=1}^{\infty} a_n \delta_{nk} \|E_k\|^2 \\ &= a_k \|E_k\|^2.\end{aligned}$$

It follows that

$$a_k = \frac{\langle f\,,\, E_k\rangle}{\|E_k\|^2}. \tag{1.41}$$

EXAMPLE: The vibrating string of variable density. This is our usual string problem, but now suppose the linear density of the string varies with x, $\rho = \rho_0\, q(x)$ say where ρ_0 is a constant and $q(x)$ a positive definite function. The differential equation is now,

$$y_{tt} = \frac{\gamma}{\rho_0 q(x)} y_{xx}.$$

Again we separate the variables, writing $y = X(x)T(t)$, so that

$$X\ddot{T} = \frac{\gamma}{\rho_0 q(x)} T X''.$$

Multiplying each side by $\rho_0/(\gamma XT)$ gives,

$$\frac{\ddot{T}\rho_0}{T\gamma} = \frac{X''}{Xq(x)} = \lambda^2. \tag{1.42}$$

The constant λ^2 must be positive, otherwise our solutions would grow exponentially with time. The Sturm-Liouville problem for X is,

$$X'' + \lambda^2 q(x) X = 0, \qquad X(0) = X(\ell) = 0. \tag{1.43}$$

. We will not be able to solve this equation unless we make a suitable choice for $q(x)$, so we take

$$q(x) = \frac{1}{s^2}, \qquad s = 1 + \frac{x}{\ell}.$$

It is now convenient to use s rather than x as the independent variable. Since,

$$\frac{\mathrm{d}x}{\mathrm{d}x} = \frac{1}{\ell}, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\ell}\frac{\mathrm{d}}{\mathrm{d}s},$$

and (1.43) transforms to

$$s^2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + \mu y = 0, \qquad \mu = \lambda^2 \ell^2, \qquad y(1) = y(2) = 0.$$

Solutions can be found by using the trial form $y = s^\alpha$. Substituting into the differential equation gives a quaratic

$$\alpha^2 - \alpha + \mu = 0$$

to determine the powers α_1, α_2. First note that for real solutions of this quadratic

$$y = As^{\alpha_1} + Bs^{\alpha_2} \tag{1.44}$$

Where A and B are constants. The boundary condition $y(1) = 0$ shows that $B = -A$ and the boundary condition $y(1) = 0$ then gives $\alpha_1 = \alpha_2$ so the solution is identically zero. We must therefore look for complex solutions of (1.44), which are given by the general formula,

$$\alpha = \tfrac{1}{2}\big(1 \pm \sqrt{1 - 4\mu}\big).$$

The solutions are complex when $\mu > \frac{1}{4}$ and are given by,

$$\alpha = \tfrac{1}{2} \pm \mathrm{i}\beta, \qquad \beta = \tfrac{1}{2}\sqrt{4\mu - 1}.$$

Since

$$s^{\mathrm{i}a} = e^{\mathrm{i}\beta \ln(s)} = \cos\big(\beta \ln(s)\big) + \mathrm{i} \sin\big(\beta \ln(s)\big)$$

the independent solutions of the equation are

$$s^{1/2} \cos\big(\beta \ln(s)\big), \qquad s^{1/2} \sin\big(\beta \ln(s)\big).$$

The solution for y is therefore,

$$y = As^{1/2} \cos\big(\beta \ln(s)\big) + Bs^{1/2} \sin\big(\beta \ln(s)$$

where A and B are arbitrary constants. The boundary condition $y(1) = 0$ implies that $A = 0$ so,

$$y = Bs^{1/2}\sin\big(\beta\ln(s)\big).$$

Now the boundary condition $y(2) = 0$ implies that

$$\beta\ln(2) = n\pi, \qquad \beta = \frac{n\pi}{\ln(2)} = \beta_n \quad \text{say}, \quad n = 1,\ 2,\ 3, \ldots.$$

Since $4\beta^2 = 4\mu - 1$ and $\mu = \lambda^2\ell^2$, the eigenvalues are

$$\lambda_n^2 = \frac{1}{\ell^2}\left(\tfrac{1}{4} + \beta_n^2\right),$$

and the corresponding eigenfunctions,

$$E_n(s) = s^{1/2}\sin\big(\beta_n\ln(s)\big),$$

or in terms of the original variable x,

$$E_n(x) = (1 + x/\ell)^{1/2}\sin\big(\beta_n\ln(1 + x/\ell)\big).$$

Equation (1.42) shows that the solution for $T(t)$ is

$$T(t) = B\cos(\lambda_n c_0 t), \qquad c_0^2 = \frac{\gamma}{\rho_0}.$$

The solution therefore takes the form,

$$y = \sum_{n=1}^{\infty} a_n E_n(x)\cos(\lambda_n c_0 t),$$

and the initial condition gives,

$$\sum_{n=1}^{\infty} a_n E_n(x) = f(x).$$

The coefficients a_n are given by the usual formula, but it is not possible to evaluate the integrals explicitly.

The figure below shows the first three eigenfunctions $E_n(x)$.

1.4 More general separation of variables problems

The partial differential equations we've solved so far were homogeneous (i.e. there was no forcing term) and the boundary conditions were homogeneous too — i.e. of the form,

$$c_1 y(a) + c_2 y'(a) = 0, \qquad c_3 y(b) + c_4 y'(b) = 0.$$

In this section we look at look at a few examples where either the pde or the boundary conditions are non-homogeneous.

EXAMPLE 1: Heat conduction in a rod with ends maintained at different temperatures.

The temperature $u(x,t)$ in a rod $0 \le x \le \ell$ satisfies the heat conduction equation

$$u_t = \kappa u_{xx}. \tag{1.45}$$

The end $x = 0$ is maintained at temperature 0, and the end $x = \ell$ at temperature u_0 (a constant). Find $u(x,t)$ given an initial temperature distribution $f(x)$.

The mathematical problem is:

$$u_t + \kappa u_{xx} = 0, \quad u(0,t) = 0, \quad u(\ell,t) = u_0, \quad u(x,0) = f(x).$$

The boundary condition at $x = \ell$ is going to cause problems. We can find separated solutions which satisfy this boundary condition, but when we add them up to obtain the full solution, the u_0's at the right-hand end will also be added, which will mess up the boundary this boundary condition. The remedy is to change the problem so that we look for a function

$$v(x,t) = u(x,t) - p(x,t) \tag{1.46}$$

where $p(x,t)$ is a particular solution of the heat equation satisfying boundary conditions

$$p(0,t) = 0, \qquad p(\ell,t) = u_0.$$

The new function $v(x,t)$ still satisfies (1.45) since it is a linear combination of solutions, but the boundary conditions are now

$$v(0,t) = 0, \qquad v(\ell,t)0,$$

i.e. they are homogeneous. The particular solution $p(x,t)$ can be as simple as we like. To make life easy we suppose p is independent of t so it represents a steady-state solution $p(x)$. Then $p(x)$ must satisfy,

$$p'' = 0, \quad p(0) = 0, \quad p(\ell) = u_0, \text{ so } p(x) = u_0 x/\ell.$$

Nice and simple! Now the problem for $v(x,t)$ is

$$v_t = \kappa v_{xx}, \quad v(0,t) = v(\ell,t) = 0, \quad v(x,0) = f(x) - u_0 x/\ell.$$

The last equation is found by setting $t = 0$ in (1.46).

Now we have a standard problem to solve we separate variables in the usual way, looking for solutions of the form $v(x,t) = X(x)T(t)$, and finding that

$$XT = \sin\left(\frac{n\pi x}{\ell}\right)\exp(-\kappa n^2\pi^2/\ell^2), \quad n = 1\,,\,2\,,\,\ldots\,.$$

Taking a combination of the separated solutions,

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)\exp(-\kappa n^2\pi^2 t/\ell^2)$$

where the a_n are constants determined by the initial condition,

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right) = f(x) - u_0 x/\ell.$$

The formula (1.41) gives,

$$a_k = \frac{2}{\ell}\,\langle \sin(\pi x/\ell)\,,\, f(x) - u_0 x/\ell\rangle.$$

Now that we've found $v(x,t)$ we have $u(x,t) = v(x,t) - u_0 x/\ell$.

EXAMPLE 2: Heat diffusion with heat sources.

Our rod $0 \le x \le \ell$ has each end fixed at temperature zero, and is heated by heat source distribution $h(x)$ per unit length. The $u(x,t)$ in the rod must satisfy,

$$u_t = \kappa u_{xx} + h(x) \tag{1.47}$$

$$u(0) = u(\ell) = 0, \qquad u(x,0) = f(x).$$

As before we make the problem homogeneous by defining

$$v(x,t) = u(x,t) - p(x,t)$$

where $p(x,t)$ is a particular solution of (1.47) satisfying the boundary conditions $p(0,t) = p(\ell,t) = 0$. For simplicity suppose that p is a function of x only, so

$$p'' = -h(x), \qquad p(0) = p(\ell) = 0.$$

This problem can be solved using a Green's function, or simply by direct integration. For example if we take $h(x) = h_0$, a constant, then

$$p(x) = \tfrac{1}{2} h_0 x(\ell - x),$$

as is easily verified. The function $v(x,t)$ must now satisfy,

$$v_t = \kappa v_{xx}, \quad v(0,t) = v(\ell,t) = 0, v(x,0) = f(x) - \tfrac{1}{2} h_0 x(\ell - x).$$

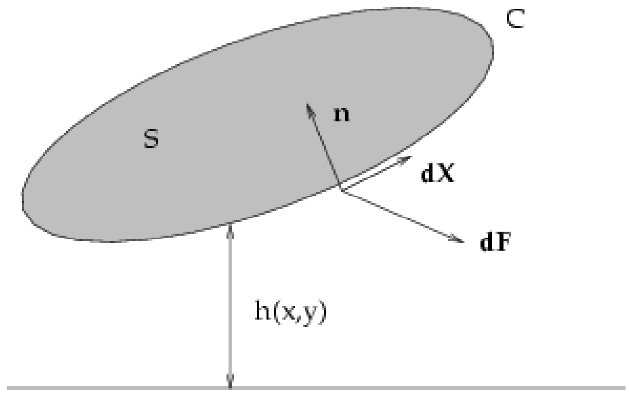
This homogeneous problem can now be solved by the usual methods.

1.5 Bessel functions

So far all our separation of variable problems involved Cartesian co-ordinates, but some problems are more naturally expressed in cyldrical polar or spherical polar co-ordinates.

The vibrating drum

The circular counterpart of the vibrating string is the drum. We consider a membrane of surface density σ which is stretched under uniform tension T. The height of the membrane above the plane $z = 0$ is $h(x,y)$ say. The diagram below represents a small section S of the membrane. The vector $\mathrm{d}\boldsymbol{x}$ is a parallel to the rim $\mathcal{C}$, $\widehat{\boldsymbol{n}}$ is normal to the membrane surface, and $\mathrm{d}\boldsymbol{F} = T\mathrm{d}\boldsymbol{x} \times \widehat{\boldsymbol{n}}$ is the external force the element $\mathrm{d}\boldsymbol{x}$ of $\mathcal{C}$.



The total force on the membrane element is

$$\int_{\mathcal{C}} \mathrm{d}\boldsymbol{F} = T \int_{\mathcal{C}} \mathrm{d}\boldsymbol{x} \times \widehat{\boldsymbol{n}}.$$

The equation of the surface of the membrane is $z = h(x, y)$ so the the vector

$$\nabla\big(z - h(x, y)\big) = (-h_x\,,\ -h_y\,,\ 1)$$

where $h_x = \partial h/\partial x$ etc. We make the same sort of assumption as we made for the vibrating string — that h_x and h_y are small, so the vector $(-h_x\,, -h_y\,, 1)$ is approximately a unit vector. Now,

$$\mathrm{d}\boldsymbol{x} \times \widehat{\boldsymbol{n}} = \begin{vmatrix} \widehat{\boldsymbol{x}} & \widehat{\boldsymbol{y}} & \widehat{\boldsymbol{z}} \\ \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \\ h_x & h_y & 1 \end{vmatrix}.$$

The z-compnent of the force is therefore given by

$$F_z = T \int_{\mathcal{C}} (h_x \mathrm{d}y - h_y \mathrm{d}x).$$

Using Green's theorem in the plane,

$$\int_{\mathcal{C}} f(x, y)\mathrm{d}x + g(x, y)\mathrm{d}y = \int_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) \mathrm{d}x\mathrm{d}y,$$

we find that

$$F_z = T \int_S \nabla^2 h \, \mathrm{d}x \, \mathrm{d}y.$$

Equating force and (mass $\times$ acceleration) per unit area of membrane we obtain the membrane equation,

$$\sigma h_{tt} = T\nabla^2 h,$$

or

$$h_{tt} = c^2 \nabla^2 h, \qquad c^2 = T/\sigma. \tag{1.48}$$

Equation (1.48) is the two-dimensional wave equation.

We consider a circular membrane under uniform surface tension T whose rim $r = a,\ z = 0$ is fixed. Here r , θ , z are cylindrical polar co-ordinates. The vertical displacement of the drum is $h(r, \theta, t)$, which must satisfy the wave equation,

$$h_{tt} = c^2 \nabla^2 h. \tag{1.49}$$

Since the rim of the drum is fixed

$$h(a, \theta, t) = 0. \tag{1.50}$$

We suppose that the initial displacement of the drum is some prescribed function, and that the initial velocity is zero.

$$h(r, \theta, 0) = f(r, \theta), \qquad \dot{h}(r, \theta, 0) = 0. \tag{1.51}$$

It is shown in appendix A that the expression for the Laplacian in polar co-ordinates is

$$\nabla^2 f = f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2}. \tag{1.52}$$

Separating the variables we write

$$f = R(r)\Theta(\theta)T(t)$$

Where $R(r)$, $\Theta(\theta)$ and $T(t)$ are functions of r θ and t only. Substituting into (1.49) and using (1.52),

$$\frac{1}{c^2}\ddot{T}R\Theta = \Theta\big(R'' + \frac{1}{r}R'\big)T + \frac{\Theta'' RT}{r^2}.$$

Now multiplying each side by $r^2/T\Theta R$ we find,

$$\frac{r^2\ddot{T}}{c^2 T} - \frac{r^2}{R}\big(R'' + \frac{1}{r}R'\big) = \frac{\Theta''}{\Theta} = -m^2, \tag{1.53}$$

since the LHS is a function of r and t only and the RHS of θ only, so both must be equal to a constant, say m^2. The function Θ satisfies the differential equation,

$$\Theta'' + m^2\Theta = 0,$$

and the general solution is,

$$\Theta = A\sin(m\theta) + B\cos(m\theta)$$

where A and B are arbitrary constants. Because the drum is circular, $\Theta(\theta)$ must be periodic with period 2π, so m must be an integer. From (1.53) it now follows that

$$\frac{\ddot{T}}{c^2 T} = \frac{1}{R}\big(R'' + \frac{1}{r}R'\big) - \frac{m^2}{r^2} = -\lambda^2$$

where λ is a constant since the RHS is a function of t only and the RHS a function of r only. This separation constant must be negative to ensure that T does not grow exponentially in time. The function R must satisfy the differential equation

$$r^2 R'' + rR' + (\lambda^2 r^2 - m^2)R = 0. \tag{1.54}$$

. This is known as *Bessel's equation* and cannot be solved in terms of elementary functions. Now $T(t)$ must satisfy,

$$\ddot{T} + \lambda^2 c^2 T = 0,$$

and the zero initial velocity condition (1.51) shows that

$$T = \cos(\lambda ct).$$

Solution of Bessel's equation

Before we can make any progress with this problem, we some information about the solutions of Bessel's equation. First note that the equation is *singular* at the origin. If we set initial values for $y(0)$ and $y'(0)$, the equation

$$y'' = \frac{(m^2 - \lambda^2 x^2)y - xy'}{x^2}$$

does not give a sensible answer for $y''(0)$, so we cannot proceed to integrate the equation numerically.

Note also that if we set $\lambda x = s$ in (1.54) then

$$\frac{\mathrm{d}}{\mathrm{d}x} = \lambda \frac{\mathrm{d}}{\mathrm{d}s},$$

and the equation becomes

$$\frac{s^2}{\lambda^2}\lambda^2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + \frac{s}{\lambda}\lambda \frac{\mathrm{d}y}{\mathrm{d}s} + (s^2 - m^2)y = 0,$$

or

$$s^2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + s\frac{\mathrm{d}y}{\mathrm{d}s} + (s^2 - m^2)y = 0. \tag{1.55}$$

Thus if $y(s)$ is a solution of (1.55), the solution of (1.54) is $y(\lambda x)$, so we may as well take $\lambda = 1$.

Our strategy is to try a power series solution,

$$y = x^\alpha \sum_{k=0}^{\infty} a_k x^k$$

of the standardised form of Bessel's equation,

$$x^2 y'' + xy' + (x^2 - n^2)y = 0. \tag{1.56}$$

We have kept an open mind and allowed the power series to begin at x^α where α is any real number. Substitiuting the power series into the Bessel's equation we obtain,

$$\sum_{k=0}^{\infty} a_k (k+\alpha)(k+\alpha-1)x^{k+\alpha} + \sum_{k=0}^{\infty} a_k (k+\alpha)x^{k+\alpha}$$

$$+ \sum_{k=0}^{\infty} a_k x^{k+2+\alpha} - m^2 \sum_{k=0}^{\infty} x^{k+\alpha} = 0.$$

Collecting together terms of like powers, we find,

$$\sum_{k=0}^{\infty} a_k \left[(k+\alpha)^2 - m^2\right] x^{k+\alpha} + \sum_{k=0}^{\infty} a_k x^{k+\alpha+2} = 0.$$

Our strategy is to find formulae for the coefficients a_k by equating like powers of x. For the lowest power x^α we find

$$a_0(\alpha^2 - m^2) = 0,$$

so $a_0 = 0$ or $\alpha = \pm m$. We reject the first possibility since we want x^α to be the lowest power of x in the solution and take $\alpha = m$ since we are looking for a solution which does not have a singularity at $x = 0$. Now equating terms in $x^{\alpha+1}$ gives,

$$a_1\left[(1+\alpha)^2 - m^2\right] = 0,$$

so $a_1 = 0$ or $\alpha = -1 \pm m$. Since we've already decided that $\alpha = m$ we reject the latter alternative and set $a_1 = 0$. Finally, equating terms in $x^{k+\alpha+2}$ we find,

$$a_{k+2}\left[(k+2+m)^2 - m^2\right] + a_k = 0,$$

or,

$$a_{k+2} = -\frac{a_k}{(k+2)(k+2+2m)}. \tag{1.57}$$

Since $a_1 = 0$ (1.57) shows that all coefficients a_k vanish if k is odd. The even coefficients are also determined by (1.57):

$$a_2 = -\frac{a_0}{(1+m)2^2}, \quad a_4 = \frac{a_0}{2!(m+1)(m+2)2^4},$$

$$a_6 = -\frac{a_0}{3!(m+1)(m+2)(m+3)2^6}.$$

The general formula is,

$$a_{2k} = \frac{(-1)^k a_0}{k!(m+1)(m+2)\dots(m+k)2^{2k}}.$$

To obtain the simplest form of solution we set

$$a_0 = \frac{1}{m!2^m}$$

which gives,

$$a_{2k} = \frac{(-1)^k}{k!(m+k)!2^{2k+m}}.$$

The corresponding solution or *Bessel function* is denoted by $J_m(x)$,

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{m+2k}.$$

A more general definition of $J_m(x)$ can be given when m is not an integer:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(m+k+1)} \left(\frac{x}{2}\right)^{m+2k}.$$

Here $\Gamma(x)$ is the *gamma function* with the property that

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(m) = (m-1)! \quad \text{if } m \text{ is an integer.}$$

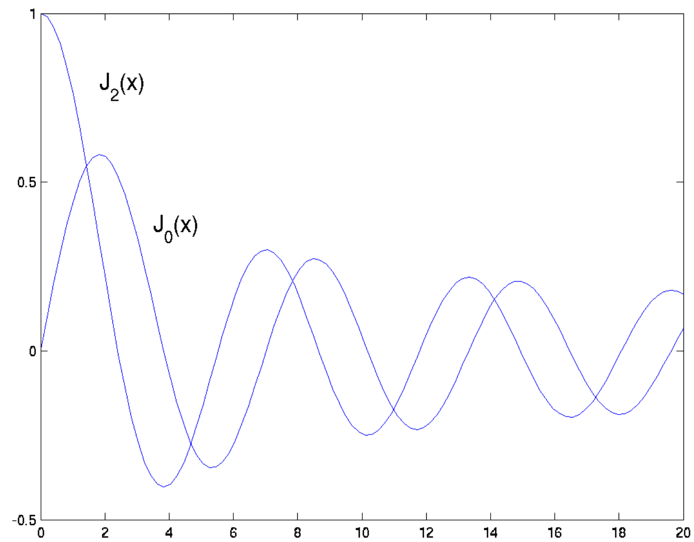
If b_k is the k'th term in the Bessel function series,

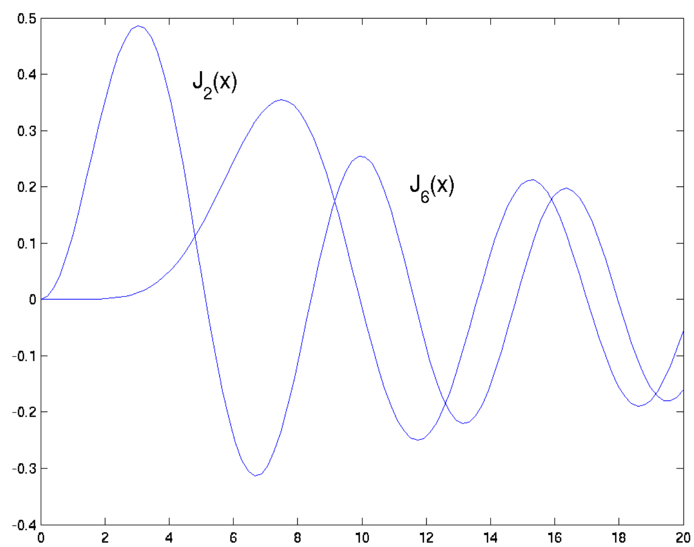
$$\frac{b_{k+1}}{b_k} = \frac{-(x/2)^2 k!(m+k)!}{(k+1)!(m+k+1)!} = \frac{-(x/2)^2}{(k+1)(m+k+1)}$$

Clearly,

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = 0,$$

for all values of x, so the Bessel function series converges for all x. (In the complex plane $J_m(x)$ is an *entire function* — i.e. it is analytic everywhere.)





Second solution of Bessel's equation

Since Bessel's equation is second-order and linear there must be a second independent solution, which we can find using the Wronskian method. Dividing (1.56) by x^2,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right) y = 0,$$

so the Wronskian is given by

$$W = A\exp\left(-\int \frac{1}{x}\,\mathrm{d}x\right) = A\exp\bigl(-\ln(x)\bigr) = \frac{1}{x}.$$

The second solution $y_2(x)$ say is given by

$$y_2(x) = J_m(x)\int\left(\frac{1}{J_m^2(x)}\frac{1}{x}\,\mathrm{d}x\right).$$

If x is very small, $J_m(x) \approx a_0x^m$ for positive m so

$$y_2(x) \approx a_0x^m\int\left(\frac{1}{a_0x^{m+1}}\,\mathrm{d}x\right) = -\frac{x^m x^{-2m}}{2ma_0} = -\frac{x^{-m}}{2ma_0}.$$

Thus the second solution behaves as x^{-m} for small x. This singularity at $x = 0$ is a consequence of the singularity in Bessel's equation. When $m = 0$ $J_0(x) \approx 1$ for small x and we find that

$$y_2(x) \approx \ln(x),$$

so again there is a singularity.

The second independent solution of Bessel's equation is usually denoted by $Y_m(x)$ If you really want to know, the formula for $Y_0(x)$ is

$$Y_0(x) = \frac{2}{\pi}\bigl[\ln(x/2)+\gamma\bigr]J_0(x) - (1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \ \ldots\ .$$

Here γ is Euler's constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right).$$

Back to the vibrating drum problem

We showed that the radial function $R(r)$ satisfied

$$r^2R'' + rR' + (\lambda^2r^2 - m^2)R = 0.$$

The solution is a Bessel function and since we don't want a singularity at $r = 0$, the centre of the drum, we must choose the J type Bessel function, $R = J_m(\lambda r)$. At the rim of the drum the displacement is zero so

$$J_m(\lambda a) = 0.$$

This is the equation which determines the eigevalues. We must have

$$\lambda_n = \frac{\xi_{nm}}{a}, \qquad n = 1,\ 2,\ \ldots$$

where ξ_{nm} is the n'th zero of $J_m(x)$. The separated solutions are therfore,

$$h = J_m(\lambda_n r)\cos(\lambda_n ct)[A\sin(m\theta) + B\cos(m\theta],$$

where m ranges from 0 to ∞ and n from 1 to ∞. Notice that our combination of separated solutions will in geneeral be a doubly infinite sum. To make things easier suppose that the drum displacement is axisymmetric, $h = h(r,t)$ with initial displacement $h(r,0) = f(r)$. Now $m = 0$, and the form of the solution is

$$h(r,t) = \sum_{k=1}^{\infty} a_k J_0(\lambda_k r)\cos(\lambda_n ct)$$

where now $\lambda_n = \xi_{k0}/a$. As usual the initial displacement will determine the coefficients a_k:

$$f(r) = \sum_{k=1}^{\infty} a_k J_0(\lambda_k r). \tag{1.58}$$

The eigenfunctions $E_k(x) = J_0(\lambda_k r)$ arise from a Sturm-Liouville problem, so we know that they are orthogona. Since the differential equation was singular though perhaps we had better check. The self-adjoint form of Bessel's equation $(n = 0)$ is

$$(xy')' + \frac{\lambda_n^2}{x} y = 0.$$

Thus two eigenfunctions $E_n(x)$, $E_m(x)$ satisfy,

$$L(E_m) = -\lambda_m^2 x E_m,$$

$$L(E_n) = -\lambda_n^2 x E_n,$$

where L is the operator $L(y) = (xy')'$. Multiplying the first equation by E_n, the second by E_m, subtracting and integrating using Green's identity we find,

$$\left[x(E_n E_m' - E_m E_n')\right]_0^1 = \int_a^b (\lambda_m^2 - \lambda_n^2) E_m E_n x \mathrm{d}x.$$

The left-hand vanishes because $x = 0$ at one end of the interval, and the eigenfunctions are zero at the other. Since $\lambda_n^2 \neq \lambda_m^2$ we conclude that

$$\int_a^b E_m E_n x \, \mathrm{d}x = 0.$$

Notice that the weight function is x.

The a_k are given by the formula,

$$a_k = \frac{\langle f(r)\,,\, E_k(r)\rangle}{\|E_k\|^2},$$

where

$$\langle f(r)\,,\, E_k(r)\rangle = \int_0^a r f(r) J_0(\lambda_k r)\mathrm{d}r, \qquad \|E_k\|^2 = \int_0^a r J_0(\lambda_k r)^2 \mathrm{d}r.$$

Unless $f(r)$ is a fairly simple function the first integral may not be expressible in terms of elementary functions, but there are relatively simple formulae for the second integral. These are given in appendix 2.

1.6 Legendre Polynomials

We now look at the problem of solving Laplace's equation

$$\nabla^2 f = 0$$

in spherical polar co-ordinates. Assume that f is axisymmetric, i.e. that there is no ϕ dependence, and Laplace's equation then takes the form

$$f'' + \frac{2f'}{r} + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) = 0. \tag{1.59}$$

Separating the variables we look for solutions of the form, $R(r)\Theta(\theta)$, and substitution into (1.59) gives

$$\left(R'' + \frac{R'}{r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\Theta_\theta\right) = 0.$$

Now we multiply each side by $r^2/(R\Theta)$ to separate the variables.

$$\frac{r^2R'' + 2rR'}{R} - = -\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\Theta_\theta\right) = \lambda \tag{1.60}$$

where λ being equal to a function of r only and to a function of θ only, must be a constant. We now make a change of variable, letting $x = \cos\theta$. In spherical polars θ ranges over the interval $[0\,,\,\pi]$ so x ranges over the interval $[0\,,\,1]$. Now,

$$\mathrm{d}x = -\sin\theta\mathrm{d}\theta, \quad \text{so} \quad \frac{\mathrm{d}}{\mathrm{d}x} = -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta},$$

So the equation for Θ takes the form,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right) + \lambda\Theta = 0.$$

Here we have written the $\sin\theta$ in the brackets in (1.60) as $\sin^2\theta/\sin\theta$ and used the identity $\sin^2\theta = 1 - x^2$. Thus $\Theta(x)$ satisfies *Legendre's equation*,

$$[(1-x^2)y']' + \lambda y = 0. \tag{1.61}$$

Notice that this equation has singularities at $x = 0$ and $x = 1$ where the coefficient of y'' vanishes.

As with Bessel's equation we have to look for a power series solution, letting

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Substituting (1.61), which can be written in the form,

$$(1-x^2)y'' - 2xy' + \lambda y = 0,$$

we obtain

$$\sum_{k=0}^{\infty} a_k \left[k(k-1)x^{k-2}(1-x^2) - 2kx^k + \lambda x^k\right] = 0.$$

Since $-k(k-1) - 2k = -k(k+1)$

$$\sum_{k=0}^{\infty}\big[(\lambda - k(k+1)\big] a_k x^k + \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = 0.$$

The second summation begins at $k = 2$ since the $k = 0$ and $k = 1$ terms vanish. We replace the dummy summation variable k in this term by $k+2$ to find,

$$\sum_{k=0}^{\infty}\Bigg[[(\lambda - k(k+1)]a_k + [(k+1)(k+2)]a_{k+2}\Bigg]x^k = 0.$$

The recursion formula for a_k is therefore

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+1)(k+2)}a_k, \quad k = 0,\ 1,\ 2,\ \ldots\ .$$

The coefficients a_0, a_1 are arbitrary constants. If we take $a_0 \neq 0$ and $a_1 = 0$ we obtain a series of even powers of x and choosing $a_1 \neq 0$ and $a_0 = 0$ give a series with only odd powers. The most interesting thing though is that if

$$\lambda = n(n+1)$$

where n is an integer we find a_{n+2} and all subsequent coefficients vanish, so the solution is a polynomial. If $\lambda \neq n(n+1)$ the power series is infinite and *fails to converge for* $x = 1$, — i.e. for $\theta = 0$. Thus if we want nonsingular solutions we must take $\lambda = n(n+1)$. After fiddling around with the recurrence relations (details are given in ..) we find we can write the solution in the form,

$$y = P_n(x) = \frac{1}{2^n} \sum_{k=0^m} \frac{(-1)^k (2n-2k)!}{k!(n-2k)!(n-k)!} x^{n-2k}, \tag{1.62}$$

where $m = \frac{1}{2}n$ if n is even and $m = \frac{1}{2}(n-1)$ in n is odd. The polynomial $P_n(x)$ is called a *Legendre polynomial*. The first few are given by,

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \dots .$$

Now we can find $R(r)$. Equation (1.60) shows that

$$R'' + 2rR' - n(n+1)R = 0.$$

This kind of equation can be solved by trying a power of x, say $R = x^\alpha$. Substituting gives,

$$\alpha(\alpha - 1) + 2\alpha - n(n+1) = 0, \qquad (\alpha - n)(\alpha + n + 1) = 0.$$

The solutions are $\alpha = n$, $\alpha = -(n+1)$ so the separated solutions of Laplace's equation are,

$$\left(Ar^n + \frac{B}{r^{n+1}} \right) P_n(\cos\theta) \tag{1.63}$$

where A and B are constants.

Steady temperature in a sphere

The steady temperature $u(r, \theta)$ in the solid sphere $r \leq a$ satisfies Laplace's equation,

$$\nabla^2 u = 0$$

and the surface $r = a$ is maintained at temperature $f(\theta)$. Find $u(r, \theta)$.

Because u satisfies Laplace's equation we try a linear combination of the separated solutions (1.63). To avoid a singularity at the centre of the sphere we must set $B = 0$,

$$u = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta).$$

where the c_n are constant coefficients. To satisfy the surface temperature condition we must have

$$\sum_{n=0}^{\infty} a_n a^n P_n(\cos\theta) = f(\theta) \qquad \text{or} \qquad \sum_{n=0}^{\infty} c_n a^n P_n(x) = g(x)$$

where $g(x) = f(\theta),\ x = \cos\theta$. The Legendre functions form an orthogonal sequence with weight function 1, and, since they arise from a Sturm-Liouville probelem, are complete. We can prove orthogonality as follows. We know that $P_n(x)$ and $P_m(x)$ satisfy Legendre's equation,

$$L(P_n) = -\lambda_n P_n,$$

$$L(P_m) = -\lambda_m P_m.$$

where the operator,

$$L(y) = [(1-x^2)y']'$$

and $\lambda_n = n(n+1)$. Multiplying the first equation by $P_m(x)$, the second by $P_n(x)$, subtracting and integrating over the interval $-1 \le x \le 1$ using Green's identity we find,

$$\Big[(1-x^2)\big(P_m(x)P_n'(x) - P_n(x)P_m'(x)\big)\Big]_{-1}^{1} = (\lambda_m - \lambda_n)\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x.$$

Since $(1-x^2)$ vanishes at both ends of the interval we conclude that

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = 0$$

if $m \neq n$. The c_n coefficients are determined from the equation

$$\sum_{n=0}^{\infty} c_n a^n P_n(x) = g(x)$$

via the usual formula;

$$c_n a^n = \frac{\langle P_n(x)\,,\, g(x)\rangle}{\|P_n\|^2}.$$

For example, suppose $f(\theta) = \cos(2\theta) = 2\cos^2\theta - 1$ or $g(x) = 2x^2 - 1 = \frac{4}{3}P_2 - \frac{1}{3}P_0$. Using the above formula we find that

$$c_n a^n = \frac{\langle P_n\,,\, \frac{4}{3}P_2 - \frac{1}{3}P_0\rangle}{\|P_n\|^2}.$$

Because of the orthogonality of the P_n the only non-zero coefficients will be c_0 and c_2. We find,

$$c_0 = \frac{\langle P_0\,,\, -\frac{1}{3}P_0\rangle}{\|P_0\|^2} = -\tfrac{1}{3},$$

and similarly, $c_2 a^2 = \frac{4}{3}$. Thus,

$$u = -\tfrac{1}{3} + \tfrac{4}{3}(r/a)^2 P_2(\cos\theta).$$

2 The Laplace Transform

2.1 Definitions and examples

The Laplace transform is a useful method for solving ordinary and partial differential equations. It works something like this. Suppose a function $f(t)$ satisfies an ordinary differential equation. We apply the Laplace transform to $f(t)$ and obtain a new function $F(p)$ which satisfies an *algebraic* equation. Generally speaking algebraic equations are easier to solve than differential equations, so with a bit of luck we can find $F(p)$. Then we undo the Laplace transform to find $f(t)$.

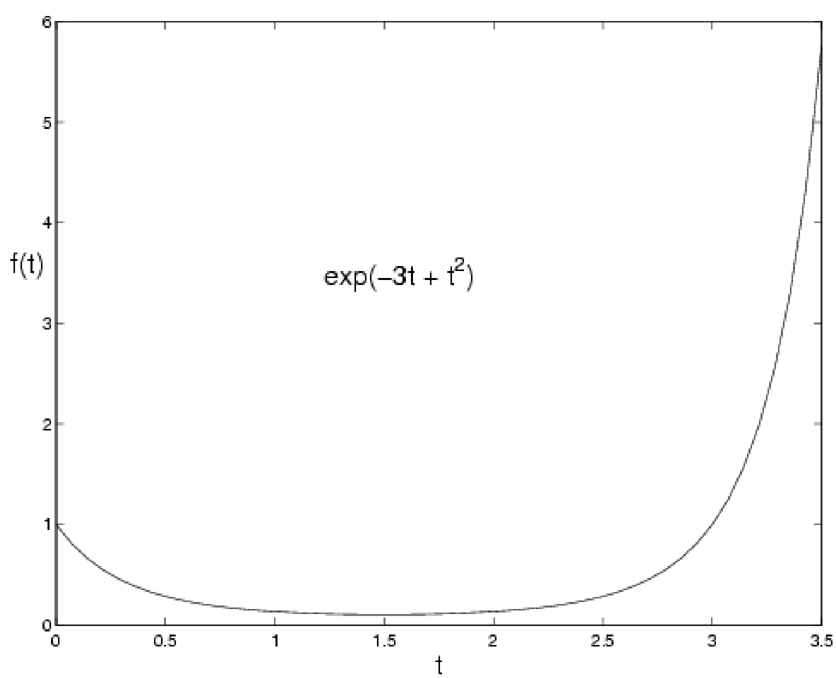
The Laplace transform of a function $f(t)$ is defined by setting

$$F(p) = \int_0^\infty e^{-pt} f(t)\mathrm{d}t.$$

This integral is *improper* in the sense that the upper limit is not finite. Such an integral is strictly speaking the limit

$$\lim_{R\to\infty} \int_0^R e^{-pt} f(t)\mathrm{d}t.$$

If the limit does not exist of course the integral is undefined. For example if $f(t) = e^{t^2}$, the graph of the integrand $e^{-pt}f(t)$ for $p = 3$ is sketched below.



The graph at first decreases but then begins to increase rapidly under the influence of e^{t^2}. The area under the graph increases without limit. An example of a convergent improper integral is

$$\lim_{R\to\infty}\int_0^R e^{-at}\mathrm{d}t = \lim_{R\to\infty}\frac{1}{a}(1-e^{-Rt}) = \frac{1}{a},$$

where a is a positive constant.

Note that the Laplace transform is linear; if $f(t)$ and $g(t)$ are functions with Laplace transforms $F(p)$ and $G(p)$, then the Laplace transform of $[af(t)+bg(t)]$ is $[aF(p)+bG(p)]$ where a and b are constants.

To find the Laplace transform of a power of t

$$\int_0^\infty e^{-pt}t^\alpha \mathrm{d}t$$

make a change of variable, $s = pt$ then

$$\int_0^\infty e^{-pt}t^\alpha \mathrm{d}t = \int_0^\infty e^{-s}\left(\frac{s}{p}\right)^\alpha \frac{\mathrm{d}x}{p} = \frac{1}{p^{1+\alpha}}\int_0^\infty e^{-s}s^\alpha \mathrm{d}s$$

$$= \frac{\Gamma(1+\alpha)}{p^{1+\alpha}}.$$

Often the symbol $\mathcal{L}$ is used to denote the operation of taking the Laplace transform. Thus we write

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(1+\alpha)}{p^{1+\alpha}}. \tag{2.1}$$

Similarly we the the symbol $\mathcal{L}^{-1}$ to denote the inverse operation, namely obtaining the function from its Laplace transform. We write

$$\mathcal{L}^{-1}\left[\frac{\Gamma(1+\alpha)}{p^{1+\alpha}}\right] = t^\alpha.$$

We're making the implicit assumption that the operation $\mathcal{L}^{-1}$ is well-defined, namely that no two functions have the same Laplace transform. We'll be able to prove this later in the course using the Fourier transform. It turns out that there is a integral formula for finding the original function from its Laplace transform.

Another common notation is to write the Laplace transform of a function $f(t)$ as $F(p)$.

Another useful Laplace transform is that of $f(t) = e^{\mathrm{i}\omega t}$ where ω is a constant.

$$F(p) = \int_0^\infty e^{-pt}e^{\mathrm{i}\omega t}\mathrm{d}t = \int_0^\infty e^{-(p-\mathrm{i}\omega t)}\mathrm{d}t = -\left[\frac{1}{p-\mathrm{i}\omega}\right]_0^\infty.$$

Thus,

$$F(p) = \frac{1}{p-\mathrm{i}\omega} = \frac{p+\mathrm{i}\omega}{p^2+\omega^2}.$$

Now taking real and imaginary parts we find,

$$\mathcal{L}(\cos\omega t) = \frac{p}{p^2+\omega^2}, \qquad \mathcal{L}(\sin\omega t) = \frac{\omega}{p^2+\omega^2}. \tag{2.2}$$

2.2 Inverting the Laplace transform

As mentioned earlier there exists an explicit formula for finding the inverse Laplace transform, but it is cumbersome to use. In many simple cases the Laplace transform can be inverted by inspection. For example, using (2.1) we can see that

$$\mathcal{L}^{-1}\left(\frac{1}{p^3}\right) = \tfrac{1}{2}t^2.$$

There are some properties of the Laplace transform which help in the inversion process. In the following formulae $F(p)$ is the Laplace transform of $f(t)$ and a is a constant.

$$\text{Rule 1:} \qquad \mathcal{L}[e^{-at}f(t)] = F(p+a).$$

This follows because

$$\int_0^\infty e^{-pt}e^{-at}f(t)\mathrm{d}t = \int_0^\infty e^{-(p+a)t}\mathrm{d}t = F(p+a).$$

$$\text{Rule 2:} \qquad \mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{p}{a}\right).$$

To prove this we make the substitution $at = s$ in the integral

$$\mathcal{L}[f(at)] = \int_0^\infty e^{-pt}f(at)\mathrm{d}t.$$

Since $t = s/a$ and $\mathrm{d}t = \mathrm{d}s/a$ the integral becomes

$$\frac{1}{a}\int_0^\infty e^{-(p/a)s}f(s)\mathrm{d}s = \frac{1}{a}F(p/a).$$

$$\text{Rule 3:} \qquad \mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}p^n} F(p).$$

This can be proved by differentiating the formula,

$$F(p) = \int_0^\infty e^{-pt} f(t)\mathrm{d}t$$

with respect to p. For example differentiating once give,

$$\frac{\mathrm{d}F}{\mathrm{d}p} = \int_0^\infty e^{-pt}(-t)f(t)\mathrm{d}t.$$

(We can differentiate under the integral sign since the integral is uniformly convergent.)

EXAMPLE 1: Find the inverse Laplace transform of

$$F(p) = \frac{1}{(a+p)^3}.$$

Note first that $\mathcal{L}(t^2) = 2/p^3$ so $\mathcal{L}^{-1}(p^{-3}) = \frac{1}{2}t^2$. Now using Rule 1,

$$f(t) = \tfrac{1}{2}e^{-at}t^2.$$

EXAMPLE 2: Find the inverse Laplace transform of

$$F(p) = \frac{3}{p^2+9}.$$

We can re-write

$$F(p) = \tfrac{1}{3}\frac{1}{(p/3)^2+1},$$

and remembering that $\mathcal{L}(\sin t) = 1/(1+p^2)$ we can apply Rule 2 to find

$$f(t) = \sin(3t).$$

2.3 Solving ordinary differential equations

But what is the purpose of finding Laplace transforms and there inverses? The useful feature of the Laplace transform is that it turns ordinary differential equations into algebraic equations, which are generally much easier to solve. Having found the Laplace transform of the solution we then invert and the problem is solved.

The key to it all is the following theorem:

Theorem: If $f(t)$ is a function with Laplace transform $F(p)$ the Laplace transform of $f'(t)$ is

$$pF(p) - f(0).$$

Proof: Integrating by parts we find that

$$\int_0^\infty e^{-pt} f'(t) \mathrm{d}t = \left[e^{-pt} f(t)\right]_0^\infty + \int_0^\infty p e^{-pt} f(t) \mathrm{d}t = pF(p) - f(0).$$

Corollary: We can easily find the Laplace transform of higher derivative by applying the formula recursively. For example,

$$\mathcal{L}(f''(t)) = p\mathcal{L}(f'(t)) - f'(0) = p\big[pF(p) - f(0)\big] - f'(0).$$

Thus

$$\mathcal{L}(f''(t)) = p^2 F(p) - pf(0) - f'(0).$$

This result can be generalised to:

$$\mathcal{L}(f^{(n)}(t) = p^n F(p) - p^{n-1)} f(0) - p^{n-2} f'(0) \ldots - p f^{n-2}(0) - f^{(n-1)}(0).$$

The following simple example illustrates the general method for finding solutions of ordinary differential equations.
EXAMPLE 1: Find the solution of the initial-value problem,

$$f'(t) = af(t), \qquad f(0) = b.$$

Taking the Laplace transform of this equation gives,

$$pF(p) - b = aF(p), \qquad F(p) = \frac{b}{p-a}$$

Using Rule 1, and the fact that $\mathcal{L}(1) = 1/p$ we can invert the Laplace transform and find

$$f(t) = be^{at}.$$

EXAMPLE 2: Find the solution of the damped simple harmonic oscillator equation,

$$\ddot{x} + 2k\dot{x} + \omega^2 x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$

Taking the Laplace transform of the equation,

$$p^2 F - p + 2k(pF - 1) + \omega^2 F = 0.$$

Solving for $F(p)$ we find

$$F(p) = \frac{2k + p}{p^2 + 2kp + \omega^2} = \frac{2k + p}{(p+k)^2 + \omega^2 - k^2}.$$

There are two cases to consider depending on the sign of $\omega^2 - k^2$. First suppose that $\omega^2 - k^2$ is positive and equal to Ω^2 say. Then

$$F(p) = \frac{2k+p}{(p+k)^2+\Omega^2} = \frac{k+p)}{(p+k)^2+\Omega^2} + \frac{k}{(p+k)^2+\Omega^2}.$$

Now inverting by inspection we find that

$$f(t) = e^{-kt}\cos(\Omega t) + \frac{k}{\Omega}e^{-kt}\sin(\Omega t). \tag{2.3}$$

Now suppose that $\omega^2 - k^2$ is negative $= -\beta^2$ say. Then

$$\begin{aligned} F(p) &= \frac{k+p}{(p+k)^2-\beta^2} \\ &= \frac{k+p}{(p+k+\beta)(p+k-\beta)} \\ &= \frac{A}{p+k+\beta} + \frac{B}{p+k-\beta}, \end{aligned}$$

where A and B are constants to be determined. To determine the constants we multiply the last equation by $(p+k+\beta)(p+k-\beta)$.

$$2k+p = A(p+k-\beta) + B(p+k+\beta).$$

This is an identity in p so we can substitute any value for p. Let $p+k=\beta$ then we find,

$$\beta + k = 2B\beta, \qquad B = \frac{\beta+k}{\beta}.$$

Now setting $p+k=-\beta$,

$$k - \beta = -2\beta A, \qquad A = \frac{\beta-k}{2\beta}.$$

Thus,

$$F(p) = \frac{1}{2\beta}\left[\frac{\beta+k}{p+k+\beta} + \frac{\beta-k}{p+k-\beta}\right].$$

Inverting we find that

$$f(t) = \frac{1}{2\beta}\left[(\beta+k)e^{-(k-\beta)t} + (\beta-k)e^{-(k+\beta)}\right]. \tag{2.4}$$

Note that (2.3) represents sub-critical damping; k is small and the solution is oscillatory. In (2.4) k is larger and the damping is supercritical. In this second case note that $k^2 = \beta^2 + \omega^2$ so $k > \beta$ and both exponential coefficients are negative.

2.4 Initial and final-value theorems

These results are sometimes useful in finding Laplace transform inverses.

Initial-value theorem:

If $\mathcal{L}f(t) = F(p)$ then,

$$\lim_{p\to\infty} pF(p) = f(0)$$

Proof:

We note first that for any function $f(t)$

$$\lim_{p\to\infty} \int_0^\infty e^{-pt} f(t)\mathrm{d}t = 0,$$

since as p increases the exponential term decreases. (This shows for example that there is no function whose Laplace transform is p.) Now

$$\mathcal{L}\big[(f'(t)\big] = \int_0^\infty e^{-pt}\big[f'(t)\big]\mathrm{d}t = pF(p) - f(0).$$

Now letting $p \to \infty$ the RHS of this equation must $\to 0$, hence the result.

Final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p).$$

Proof:

$$\int_0^\infty e^{-pt}\big[f'(t)\big]\mathrm{d}t = pF(p) - f(0).$$

The limit of the LHS as $p \to 0$ is

$$\int_0^\infty f'(t)\mathrm{d}t = f(\infty) - f(0) = \lim_{p\to\infty} pF(p) - f(0).$$

Thus, cancelling $f(0)$ from each side gives

$$\lim_{t\to\infty} (f(t) = \lim_{p\to 0} pF(p).$$

2.5 Convolution product

One of the most useful methods for inverting Laplace transforms is given by the following theorem:

Theorem: If $f(t)$ and $g(t)$ are functions with Laplace transforms $F(p)$ and $G(p)$ then,

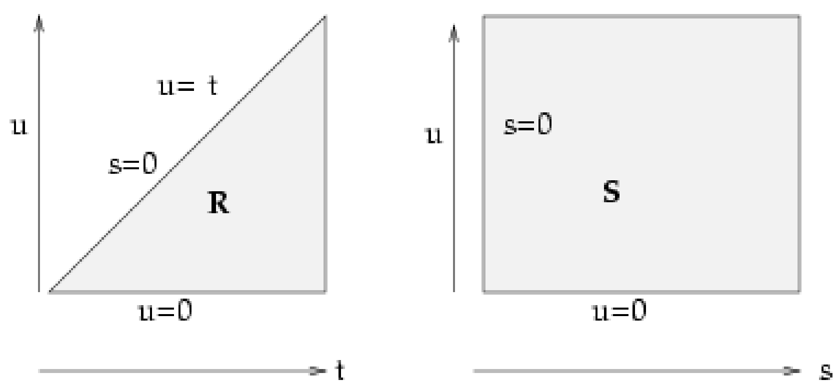
$$\mathcal{L}^{-1}(F(p)G(p)) = \int_0^t f(t-u)g(u)\mathrm{d}u.$$

The integral on the RHS is called the *convolution product* of $f(t)$ and $g(t)$, and is written as $f * g$.

Proof: To prove the result we need to show that $\mathcal{L}(f * g) = F(p)G(p)$.

$$\mathcal{L}(f * g) = \int_0^\infty e^{-pt}\left[\int_0^t f(u)g(t-u)\mathrm{d}u\right]\mathrm{d}t.$$

This is an integral over the region R in the (t,u) plan shown in the figure on the following page.



We transform the t variable in the integration to $s = t-u$, so the integration is now over region S in the figure. The Jacobian of the transformation is 1, therefore

$$\mathcal{L}(f * g) = \int_S e^{-p(s+u)}f(u)g(s)\mathrm{d}u\mathrm{d}s$$

$$= \left(\int_0^\infty e^{-ps}f(s)\mathrm{d}s\right)\left(\int_0^\infty e^{-pu}f(u)\mathrm{d}u\right) = F(p)G(p).$$

This completes the proof.

Note that the convolution product is commutative — i.e. $f * g = g * f$. This follows of course since the Laplace transform $F(p)G(p)$ is commutative. It can also be shown directly by making the substitution $s = t - u$ in the definition of $f * g$.

EXAMPLE 1: Find the solution of the initial-value problem,

$$\ddot{x} + \omega^2 x = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0,$$

where ω is a constant and $f(t)$ some forcing function. Let $\mathcal{L}\big(x(t)\big) = X(p)$. Taking the Laplace transform we find,

$$p^2 X + \omega^2 X = F(p)$$

where $F(p)$ is the Laplace transform of $f(t)$. Solving for X we obtain,

$$X = \frac{1}{p^2 + \omega^2}\, F(p).$$

The inverse Laplace transform of the first factor on the RHS is $(1/\omega)\sin(\omega t)$, so we can use the convolution theorem invert:

$$x(t) = \frac{1}{\omega} \int_0^t \sin \omega(t-u) f(u) \mathrm{d}u. \tag{2.5}$$

This formula has the advantage that it provides a solution of the forced simple harmonic oscillator for *any* function $f(t)$.

Example 2: Solve the initial-value problem,

$$\ddot{x} = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0.$$

This describes the motion of a particle of unit mass being accelerated by a time-dependent force $f(g)$. Taking the Laplace transform of the equation we fin,

$$p^2 X = F(p), \qquad X = \frac{1}{p^2} F(p).$$

Since the inverse Laplace transform of $1/p^2$ is t we find

$$x(t) = \int_0^t \big[(t-u) f(u)\big] \mathrm{d}u.$$

For example if we take $f(t) = f_0$ (a constant) we find

$$x(t) = f_0 \int_0^t (t-u)\mathrm{d}u = f_0 \left[tu - \tfrac{1}{2}u^2\right]_{u=0}^{u=t} = f_0(t^2 - \tfrac{1}{2}t^2) = \tfrac{1}{2} f_0 t^2.$$

2.6 Green's functions

In this section we look at the convolution product from a slightly different angle. First we need to introduce the strange function $\delta(t)$, which is defined as follows:

$$\delta(t) = 0 \quad \text{if} \quad t \neq 0, \qquad \int_\infty^\infty \delta(t)\mathrm{d}t = 1.$$

No such function exists in the classical sense, because if a function is non-zero at only one point, its integral must be zero. However $\delta(t)$ is an example of a "generalised " function – defined by its properties under integration with a suitable class of test functions. Note that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = \int_{-\infty}^{\infty} f(0)\delta(t)\mathrm{d}t$$

since the only contribution to the integral arises from the point $t = 0$, and it follows that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\mathrm{d}t = f(0)\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = f(0).$$

You can think of the graph of $\delta(t)$ as consisting of an isolated spike at the origin.

We consider now equations of the form,

$$L\big(x(t)\big) = f(t)$$

where L is a linear differential operator. We'll assume zero initial contitions $x(0) = \dot{x}(0) = 0$. For example we can write the forced SHM equation

$$\ddot{x} + \omega^2 x = f(t)$$

in the form

$$L\big(x(t)\big) = f(t)$$

where the differential operator L is given by

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2.$$

Because these equations are linear, solutions can be superposed — i.e. if say

$$L\big(x_1(t)\big) = f_1(t) \quad \text{and} \quad L\big(x_2(t)\big) = f_2(t)$$

it follows that the solution of

$$L(x) = \alpha f_1(t) \quad \text{is} \quad \alpha x_1(t)$$

(α is a scalar). Similarly the solution of

$$L(x) = f_1(t) + f_2(t) \quad \text{is} \quad x_1(t) + x_2(t).$$

The Green's function method provides a scheme for superposing such solutions in order to obtain a solution of

$$L\big(x(t)\big) = f(t)$$

for *any* RHS function $f(t)$. The method is based on first solving the basic equation

$$L\big(x_g(t)\big) = \delta(t).$$

We call the solution $x_g(t)$ the "Green's function" of the operator L. We now look for a suitable linear combination of delta functions to produce an $f(t)$ on the RHS. Such a combination is given by the formula

$$f(t) = \int_0^t f(t-u)\delta(u)\mathrm{d}u.$$

Thus to produce a solution of

$$L\big(x(t)\big) = f(t)$$

we take the corresponding linear combination of Green's functions, namely

$$\int_0^t f(t-u)x_g(u)\mathrm{d}u. \tag{2.6}$$

For example the Green's function for the forced SHM equation is the solution of

$$\ddot{x} + \omega^2 x = \delta(t). \tag{2.7}$$

Now the Laplace transform of $\delta(t)$ is

$$\int_0^\infty e^{-pt}\delta(t)\mathrm{d}t = 1,$$

so taking the Laplace transform of (2.7) gives

$$(p^2+\omega^2)X = 1, \quad X = \frac{1}{p^2+\omega^2} \quad , \quad x_g = \frac{1}{\omega}\sin(\omega t.$$

Substituting into (2.6) we find that the solution of

$$\ddot{x} + \omega^2 x = f(t)$$

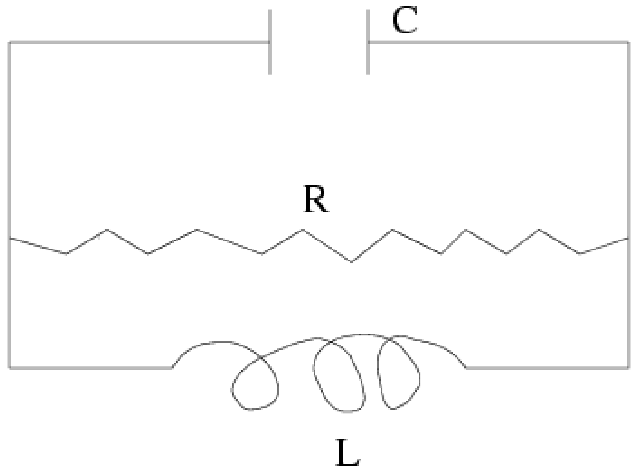
is given by

$$\frac{1}{\omega}\int_0^t \sin(\omega u)f(t-u)\mathrm{d}u.$$

This is identical to the solution (2.5) found by direct application of the Laplace transform.

2.7 Systems of equations

The Laplace transform can also be used to solve systems of equations.



The diagram represents a circuit including a capacitor, C, a resistor R and and inductor L. It can be shown (see p.345 of the textbook) that the electric current I and voltage drop V across the capacitor, satisfy the following system of equations.

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{V}{L}.$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{I}{C} - \frac{V}{RC}.$$

Suppose that $I(0) = 0$ and $V(0) = v_0$. Setting $\mathcal{L}I = \bar{I}$ and $\mathcal{L}\bar{V}$ we find that the Laplace transform of the system is

$$p\bar{I} = \frac{\bar{V}}{L}$$

$$\bar{V}\left(p + \frac{1}{RC}\right) + \frac{\bar{I}}{RC} = v_0.$$

The system of equations can be written in matrix form,

$$\begin{pmatrix} p & -L^{-1} \\ C^{-1} & p+\gamma \end{pmatrix} \begin{pmatrix} \bar{I} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} 0 \\ v_0 \end{pmatrix}.$$

where the constant $\gamma = 1 + 1/RC$. The inverse of the 2×2 matrix is

$$\frac{1}{p(p+\gamma)+\beta}\begin{pmatrix} p+\gamma & L^{-1} \\ -C^{-1} & p \end{pmatrix}$$

where $\beta = 1/(LC)$. Thus,

$$\begin{pmatrix} \bar{I} \\ \bar{V} \end{pmatrix} = \frac{1}{p(p+\gamma)+\beta}\begin{pmatrix} p+\gamma & L^{-1} \\ -C^{-1} & p \end{pmatrix}\begin{pmatrix} 0 \\ v_0 \end{pmatrix}.$$

Thus, for example,

$$\bar{V} = \frac{pv_0}{p(p+\gamma)+\beta}$$

and completing the square on the denominator gives

$$\bar{V} = \frac{pv_o}{(p+\frac{1}{2}\gamma)^2+\omega^2}, \quad \omega^2 = \beta - \tfrac{1}{4}\gamma^2.$$

Assuming that $\omega^2 > 0$ we can invert by inspection to find

$$V = v_o \exp(-\tfrac{1}{2}\gamma t)\cos(\omega t).$$

The solulution is therefore a damped oscillation.

2.8 Laplace transform of $J_0(x)$

EXAMPLE 3: Find the Laplace transform of $J_0(x)$. This function satisfies the differential equation,

$$xy'' + y' + xy = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Now we make use of Rule 3

$$\text{Rule 3:} \qquad \mathcal{L}[t^n f(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}p^n} F(p).$$

This can be proved by differentiating the formula,

$$F(p) = \int_0^\infty e^{-pt} f(t)\mathrm{d}t$$

with respect to p. For example differentiating once give,

$$\frac{\mathrm{d}F}{\mathrm{d}p} = \int_0^\infty e^{-pt}(-t)f(t)\mathrm{d}t.$$

Let $\mathcal{L}y(x) = Y(p)$. Then taking the Laplace transform of the equation and using Rule 3 we find,

$$-\frac{\mathrm{d}}{\mathrm{d}p}(p^2Y - p) + pY - 1 - \frac{\mathrm{d}Y}{\mathrm{d}p} = 0.$$

Rearranging gives,

$$(p^2+1)\frac{\mathrm{d}Y}{\mathrm{d}p} + pY = 0, \qquad \frac{\mathrm{d}Y}{Y} = -\frac{p\mathrm{d}p}{1+p^2}.$$

Integrating with respect to p we find,

$$\ln(Y) = -\tfrac{1}{2}\ln(1+p^2) + c$$

where c is a constant. Thus

$$Y = \frac{A}{(1+p^2)^{1/2}}.$$

where A is a constant. Now,

$$\mathcal{L}(y'(x)) = \int_0^\infty e^{-px}y'(x)\mathrm{d}x = pF(p) - f(0).$$

Now we apply the Final-value theorem As $p \to \infty$ the integral $\to 0$.

$$\lim_{p\to\infty} pF(p) = f(0),$$

so,

$$\lim_{p\to\infty}\frac{Ap}{(1+p^2)^{1/2}} = A = J_0(0) = 1.$$

Thus $A = 1$ and

$$\mathcal{L}(J_0(x)) = \frac{1}{(1+p^2)^{1/2}}.$$

2.9 Applications to partial differential equations

EXAMPLE 2: The temperature $u(x,t)$ in the semi-infinite medium $x > 0$ ratifies the heat diffusion equation,

$$u_t = \kappa u_{xx}.$$

The face $x = 0$ is maintained at temperature $f(t)$ and $u(x,0) = 0$. Find $u(x,t)$.

Let $U(x,p)$ be the Laplace transform (in t) of $u(x,t)$ — i.e.

$$U(x,p)=\int_0^\infty e^{-pt}u(x,t)\mathrm{d}t.$$

Then taking the Laplace transform of the diffusion equation we find,

$$pU=\kappa U_{xx},\qquad U_x x=\frac{p}{\kappa}U$$

The general solution of this differential equation is

$$U=Ae^{\sqrt{xp/\kappa}}+Be^{-x\sqrt{p/\kappa}}$$

where A and B are arbitrary constants. To avoid an unphysical solution growing exponentially with x we must choose $A=0$. Since $u(0,t)=f(t)$

$$U(0,p)=B=F(p),\qquad U(x,p)=e^{-x\sqrt{p/\kappa}}F(p).$$

Now

$$\mathcal{L}^{-1}\left(e^{-a\sqrt{p}}\right)=\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}.$$

(see problem 4 of Example sheet V.) Using the convolution theorem,

$$u(x,t)=f(t)*\frac{x}{2\sqrt{\pi\kappa t^3}}e^{-x^2/4kt}=\int_0^t\frac{x}{2\sqrt{\pi\kappa u^3}}e^{-x^2/4\kappa u}f(t-u)\mathrm{d}u.$$

For example, suppose that $f(t)$ is constant $=u_0$ say. Then

$$u_0\int_0^t\frac{x}{2\sqrt{\pi\kappa u^3}}e^{-x^2/4\kappa u}\mathrm{d}u.$$

Make the substitution,

$$\frac{x^2}{4\kappa u}=y^2,\qquad y=\frac{x}{2\sqrt{\kappa u}}=\frac{x}{2\sqrt{\kappa}}u^{-1/2}.$$

then

$$\mathrm{d}y=\frac{x}{2\sqrt{\kappa}}(-1/2)u^{(-1/2)}\,\mathrm{d}u=\frac{-x}{4\sqrt{\kappa u^3}}\mathrm{d}u.$$

Now

$$\begin{aligned}u(x,t)&=u_0\int_{x/2\sqrt{\kappa t}}^\infty\frac{x}{2\sqrt{\pi\kappa u^3}}\,\frac{4\sqrt{\kappa u^3}}{x}\,\mathrm{d}u\\&=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{\kappa t}}^\infty e^{-y^2}\,\mathrm{d}y.\end{aligned}$$

This last integral can be expressed in terms of the error function,

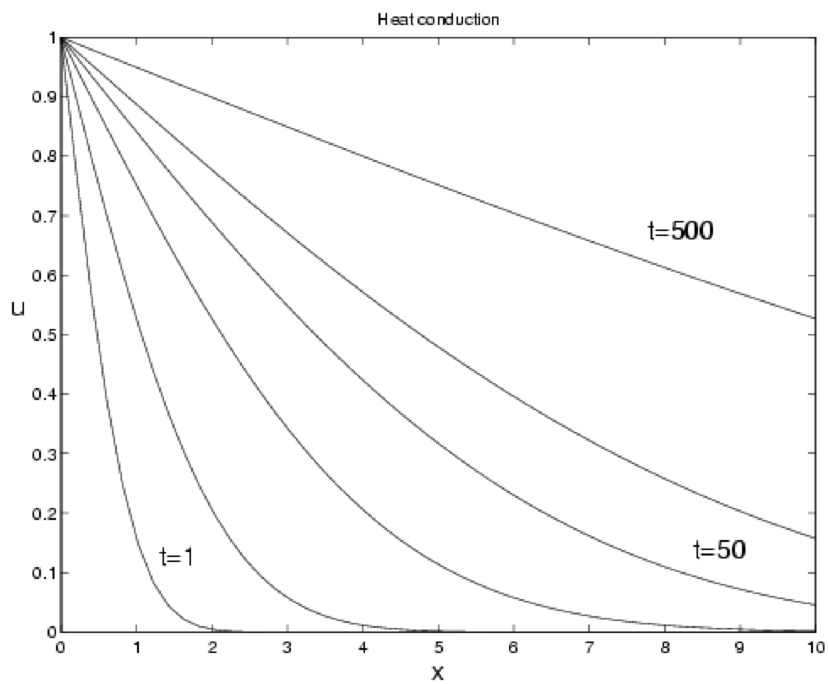
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t.$$

Since $\operatorname{erf}(x) \to 1$ as $x \to \infty$,

$$1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t.$$

The function $1 - \operatorname{erf}(x)$ is called the *complementary error functions* and is written $\operatorname{cerf}(x)$. Thus we can write our solution in the form,

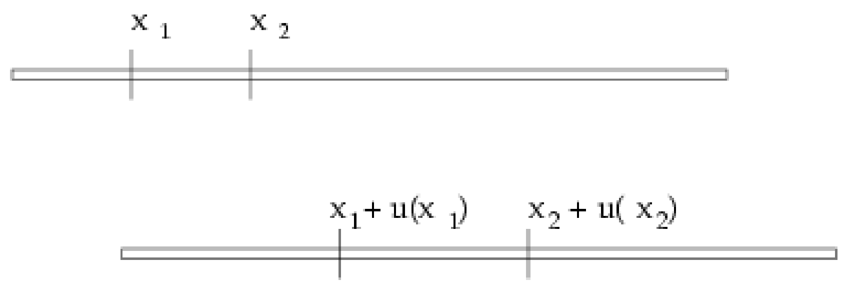
$$u(x,t) = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right).$$



2.10 Other examples

In this final section on the Laplace transform we look at two further applications.

The first concerns compressional waves in a uniform elastic rod of length ℓ.



In the diagram $u(x,t)$ is the the displacement of a particle of the rod initially and position x. The tension T due to stretching of a section say $[x_1\,,\,x_2]$ is proportional to the increase in length divided by the original length.

$$T(x) = EA\frac{u(x_2) - u(x_1)}{x_2 - x_1},$$

where A is the rod cross-section and E is Young's modulus for the material. In the limit as $x_2 \to x_1$ we find

$$T(x) = EA\frac{\partial u}{\partial x}. \tag{2.8}$$

The rate of change of momentum of the section $[0\,,\,x]$ of the rod is

$$\int_0^x A\rho u_{tt}\mathrm{d}x = T(x) - T(0),$$

the RHS being the total force acting on the rod. Differentiating with respect to x we find,

$$A\rho u_{tt} = T'(x).$$

Then using (2.8),

$$u_{tt} = c^2 u_{xx}, \qquad c^2 = E/\rho. \tag{2.9}$$

Suppose that at time $t = 0$ the rod is at rest, then the end $x = \ell$ is moved with uniform speed V. To determine $u(x,t)$ we take the Laplace transform of (2.9),

$$p^2 U = c^2 U_{xx}.$$

The general solution of this equation is

$$U = A\cosh(px/c) + B\sinh(px/c)$$

where A and B are arbitrary constants. The end $x = 0$ of the rod is free so the tension must vanish at this point and according to (2.8) $u_x(0,t) = 0$. This implies that $B = 0$. The displacement of the end $x = \ell$ is Vt, so at this point $U = V/p^2$ and

$$U(x,p) = \frac{V}{p^2}\,\frac{\cosh(px/c)}{\cosh(p\ell/c)}.$$

Suppose we want to determine the motion of the end $x = 0$. The Laplace transform of the velocity $u_t(0,t)$ of this end is

$$pU(0,p) = \frac{V}{p}\frac{1}{\cosh(p\ell/c)}.$$

Now, how do we determine the Laplace transform? First we consider the Laplace transform of the shifted Heaviside function $H(t-a)$, where a is a constant. The Laplace transform is

$$\int_0^\infty e^{-pt}H(t-a)\mathrm{d}t = \int_a^\infty e^{-pt}\mathrm{d}t = p^{-1}e^{-ap}.$$

Now,

$$\begin{aligned}\frac{V}{p}\frac{1}{\cosh(p\ell/c)} &= \frac{V}{p}\frac{2}{e^{p\ell/c}+e^{-p\ell/c}}\\ &= \frac{2V}{p}e^{-p\ell/c}\big[1+e^{-2pl/c}\big]^{-1}.\end{aligned}$$

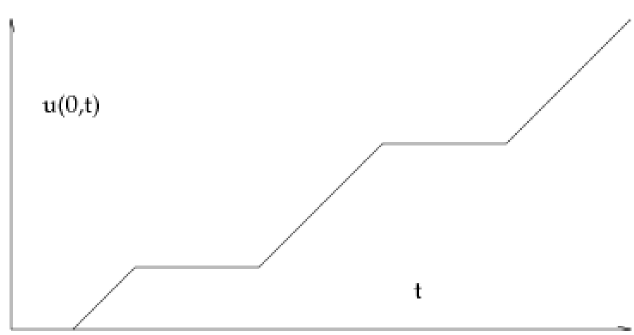
Expanding the term in square brackets as an infinite series we find,

$$\mathcal{L}[u_t(0,t)] = 2V\big[p^{-1}e^{-p\ell/c} - p^{-1}e^{-3p\ell/c} + p^{-1}e^{-5p\ell/c}\dots\big].$$

Inverting the Laplace transform we find,

$$u(0,t) = 2V\big[H(t-\ell/c) - H(t-3\ell/c) + H(t-5\ell/c)\dots\big].$$

Over the time interval $[0\,,\,\ell/c]$ the end of the rod remains stationary. This is to be expected because the disturbance at the end $x = \ell$ will take a time ℓ/c to propagate to $x = 0$. Over the time interval $[\ell/c\,,\,3\ell/c]$ the end moves with speed $2V$ in order to catch up; but over the interval $[3\ell/c\,,\,5\ell/c]$ the first two Heavisde functions cancel, and the end $x = 0$ remains stationary once more. The graph of $u(0,t)$ versus t is shown below.



2 The Fourier transform

2.1 Generalisation of Fourier Series

The idea of a Fourier transform develops naturally from the Fourier series. We assume the basic Fourier theorem, that a function $f(t)$ can be expanded as a Fourier series in the interval $[-\pi\,,\,\pi]$ in the form,

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n)\cos(nt) + b_n\sin(nt) \tag{2.1}$$

where

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(t)\cos(t)\mathrm{d}t, \quad b_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(t)\sin(t)\mathrm{d}t. \tag{2.2}$$

We need to generalise this formula to encompass expansions over intervals of any width. Let c be a constant. Then using (2.1) we see that

$$f(\frac{ct}{\pi}) = \frac{1}{1}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt), \tag{2.3}$$

where

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(\frac{ct}{\pi})\cos(t)\mathrm{d}t, \quad b_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(\frac{ct}{\pi})\sin(t)\mathrm{d}t.$$

Making the substitutions $x = ct/\pi$, $t = \pi x/c$, the integrals become

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)\mathrm{d}x, \qquad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)\mathrm{d}x. \tag{2.4}$$

Equation (2.3) can now be written in the form,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{c}\right) + b_n\sin\left(\frac{n\pi x}{c}\right). \tag{2.5}$$

The range of validity of the expansion is now $[-c\,,\,c]$.

2.2 The Fourier integral theorem

We begin by noting that by using (2.4) we can write

$$a_n\cos\left(\frac{n\pi x}{c}\right) + b_n\sin\left(\frac{n\pi x}{c}\right)$$

as

$$\frac{1}{c}\int_{-c}^{c} f(s)\cos\left(\frac{n\pi s}{c}\right)\cos\left(\frac{n\pi x}{c}\right) + \sin\left(\frac{n\pi s}{c}\right)\sin\left(\frac{n\pi x}{c}\right).$$

Then using the identity,

$$\cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A - B)$$

and (2.4) we can re-write (2.5) in the form,

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(s)\mathrm{d}s\ + \sum_{n=1}^{\infty}\int_{-c}^{c} f(s)\cos\left[\frac{n\pi}{c}(s-x)\right]\mathrm{d}s. \tag{2.6}$$

Now imagine what happens as $c \to \infty$. Assuming that the integral

$$\int_{-\infty}^{\infty} f(s)\mathrm{d}s$$

converges, we can ignore the first term of the Fourier series and write

$$f(x) = \frac{1}{c}\sum_{n=1}^{\infty}\int_{-c}^{c} f(s)\cos\left[\frac{n\pi}{c}(s-x)\right]\mathrm{d}s. \tag{2.7}$$

We define a new variable α, $0 \leq \alpha \leq \infty$ and divide the α-axis into small intervals

$$\Delta\alpha = \pi/c.$$

Since $n\pi/c = n\Delta\alpha = \alpha$ we can write (2.7) as

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-c}^{c} f(s)\cos[\alpha - x]\Delta\alpha.$$

Now, as we let

$$c \to \infty \qquad \Delta\alpha \to 0,$$

and the sum over α becomes an integration.

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\mathrm{d}\alpha\int_{-\infty}^{\infty} f(s)\cos\left[\alpha(s-x)\right]\mathrm{d}s. \tag{2.8}$$

.

2.3 The Fourier transform

Since the cosine is an even function we can re-write (2.8) in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\int_{-\infty}^{\infty} f(s)\cos\left[\alpha(s-x)\right]\mathrm{d}s. \tag{2.9}$$

Moreover, since the sine is an odd function,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} f(s)\sin\left[\alpha(s-x)\right]\mathrm{d}s = 0.. \tag{2.10}$$

Taking the sum, (2.9) minus i $\times$ (2.10) we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} f(s)\exp\bigl[-\mathrm{i}\alpha(x-s)\bigr]\mathrm{d}s.$$

Separating the exponential factors, distributing factors of $2\pi^{-1/2}$ in an even-handed manner, we can rewrite the above equation in the form,

$$f(x) = \int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(s)e^{-\mathrm{i}\alpha x}\,\frac{1}{\sqrt{2\pi}}\,e^{\mathrm{i}\alpha s}\,\mathrm{d}s. \tag{2.11}$$

We define the *Fourier transform* $\hat{f}(\alpha)$ of $f(x)$ by setting

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}\alpha s} f(s)\mathrm{d}s. \tag{2.12}$$

It follows from (2.11) that

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)e^{-\mathrm{i}\alpha x}\mathrm{d}\alpha. \tag{2.13}$$

Thus the Fourier transform comes pre-packaged with an inversion formula.

Often we use k rather than α as the Fourier transform variable, so that the Fourier transform and its inverse are written in the form,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mathrm{i}kx} f(s)\mathrm{d}x. \tag{2.14}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)e^{-\mathrm{i}kx}\mathrm{d}k. \tag{2.15}$$

We often write $\hat{f}(\alpha)$ as $\mathcal{F}\,(f)$ and $f(x)$ as $\mathcal{F}^{-1}\,(\hat{f}(k)$.

It is a simple matter to find the transform of a derivative. If we differentiate (2.15) with respect to x we find

$$f'(x) = -\mathrm{i}k\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)e^{-\mathrm{i}\alpha x}\mathrm{d}\alpha.$$

Alternatively we can express this result as

$$\mathcal{F}\,\left[f'(x)\right] = -\mathrm{i}k\mathcal{F}\,(x).$$

Example 1: Find the Fourier transform of the function $f(x) = e^{-a|x|}$ where a is a positive constant.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx} e^{-a|x|} \mathrm{d}k$$

Because of the absolute value term we must treat this as two separate integrals:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{\mathrm{i}kx} e^{-ax} \mathrm{d}x + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{\mathrm{i}kx} e^{ax} \mathrm{d}x. \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(x(\mathrm{i}k-a)}}{\mathrm{i}k - a} \right]_0^{\infty} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(x(\mathrm{i}k+a)}}{\mathrm{i}k + a} \right]_{\infty}^{0}. \end{aligned}$$

Thus,

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{\mathrm{i}k - a} + \frac{1}{\mathrm{i}k + a} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{\mathrm{i}k + a + a - \mathrm{i}k}{a^2 + k^2} = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + k^2}. \end{aligned}$$

Taking the inverse transform, we find that

$$e^{-a|x|} = \frac{2a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}kx}}{a^2 + k^2} \mathrm{d}k.$$

Example 2: The wave equation for a stretched string is

$$y_{tt} = c^2 y_{xx} \tag{2.16}$$

where $y(x,t)$ is the transverse displacement of the string and c is the wave speed. Here we consider an infinite string with initial velocity zero and initial displacement $f(x)$ say. Taking the Fourier transform of (2.16) gives

$$\hat{y}_{tt} = -c^2 k^2 \hat{y}.$$

The solution is

$$\hat{y} = A \cos(ckt) + B \sin(ckt)$$

where A and B are arbitrary constants. Taking the Fourier transform of the initial conditions

$$y(x,0) = f(x), \qquad y_t(x,0) = 0$$

shows that $B = 0$ and $A = \hat{f}(k)$, so

$$\hat{y} = \cos(ckt)\hat{f}(k).$$

The inversion formula gives

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tfrac{1}{2} e^{-\mathrm{i}kx} \left(e^{\mathrm{i}kct} + e^{-\mathrm{i}kct}\right) \hat{f}(k) \mathrm{d}k.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tfrac{1}{2} \left(e^{-\mathrm{i}k(x-ct)} + e^{-\mathrm{i}k(x+ct)}\right) \hat{f}(k) \mathrm{d}k$$

$$\tfrac{1}{2} \left(f(x-ct) + f(x+ct)\right).$$

The initial profile divides into two similar disturbances — one moving to the right and the other to the left. This solution is known as D'Alembert's solution, after the French mathematician Jean Le Rond, D'Alembert.

2.4 Fourier sine and cosine transforms

Suppose that $f(x)$ is an even function — i.e. $f(-x) = f(x)$. The Fourier transform of $f(x)$ is

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx} f(x) \mathrm{d}x. \tag{2.17}$$

If we make substitution $x = -\xi$ in this integral we find

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{-\mathrm{i}k\xi} f(-\xi)(-\mathrm{d}\xi).$$

The minus sign allows us to switch the limits of integration, and since $f(-\xi) = f(\xi)$,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}k\xi} f(\xi)(-\mathrm{d}\xi).$$

Since $f(\xi)$ is an even function we can confine the integration range to the interval $[0\,,\,\infty]$, multiplying the answer by two, to give

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}kx} f(x) \mathrm{d}x, \tag{2.18}$$

where we have reverted to x as the dummy variable of integration. Now adding equations (2.17) and (2.18) and dividing by two,

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \left[e^{-\mathrm{i}kx + e^{-\mathrm{i}kx}}\right] f(x) \mathrm{d}x.$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(kx) f(x) \mathrm{d}x. \tag{2.19}$$

This form of Fourier transform is called the *Fourier cosine transform.* We write the Fourier cosine transform of $f(x)$ as $\hat{f}_c(k)$, or $\mathcal{F}_c\, f(x)$. Similar aguments show that the inverse transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) \hat{f}_c(k) \mathrm{d}k.$$

Thus the inversion formula is exactly the same as the transform formula.

Similarly for an odd function we can define the *Fourier sine transform*

$$\mathcal{F}_s\, f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) f(x) \mathrm{d}x.$$

with inverse

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) \hat{f}_c(k) \mathrm{d}k.$$

These transforms can be used to solve partial differential equations on a semi-infinite region, so the restriction that the functions be even or odd is not an issue. For example, if we are solving say for $f(x)$ in the range $0 \le x \le \infty$ we can define values of $f(x)$ in the region $-\infty < x \le 0$ so as to make $f(x)$ and odd function.

Sine and cosine transforms of derivatives

Taking the cosine or since transform of a derivative is not quite as straightforward as for the full Fourier transform, because the transform region has an end. For example

$$\mathcal{F}_c\, f'(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos(kx) \mathrm{d}x.$$

Integrating by parts gives

$$\mathcal{F}_c\, f'(x) = \sqrt{\frac{2}{\pi}} \big[f(x) \cos(kx) \big]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)(-k \sin(x)) \mathrm{d}x.$$

Note that we assume $f(x) \to 0$ as $x \to \infty$, so

$$\mathcal{F}_c\, f'(x) = k \mathcal{F}_s\,(f) - \sqrt{\frac{2}{\pi}} f(0). \tag{2.20}$$

. A similar argument shows that

$$\mathcal{F}_s\, f'(x) = -k \mathcal{F}_c\,(f) \tag{2.21}$$

.

Of course in many applications we need double derivatives, but we can use the results already obtained. For exampe, to find $\mathcal{F}_c f''(x)$we substitute f' into the formula for $\mathcal{F}_c f'$, and find that

$$\begin{aligned}\mathcal{F}_c f'' &= k\mathcal{F}_s f' - \sqrt{\frac{2}{\pi}}f'(0) \\ &= -k^2\mathcal{F}_c f - \sqrt{\frac{2}{\pi}}f'(0).\end{aligned} \tag{2.22}$$

Using a similar argument we find

$$\mathcal{F}_s(f'') = -k^2\mathcal{F}_s(f) + k\sqrt{\frac{2}{\pi}}f(0). \tag{2.23}$$

Example: The semi-infinite region $0 \leq x \leq \infty$ is initially at temperature $f(x)$, where f is some specified function. The boundary $x = 0$ is thermally insulating. Find the temperature in the region as a function of time.

The temperature $u(x,t)$ at position x and time t satisfies the heat conduction equation,

$$u_t = \kappa u_x x.$$

For simplicity we take the thermal diffusivity $\kappa = 1$, so

$$u_t = u_x x. \tag{2.24}$$

To solve this partial differential equation we take a Fourier cosine transform of $u, x, t)$ with respect to x. The reason for choosing the cosine transform is the condition $u_x(0, t = 0$ which is satisfied by $\cos(kx)$ but not by $\sin(kx)$. Since $u_x(0, t = 0$ the cosine transform of (2.24) gives

$$U_t = -k^2 U, \quad \text{so} \quad U = Ae^{-k^2 t}$$

where A is an arbitrary constant (or more strictly speaking an arbitrary function of k. This constant is determined by the initial condition,

$$u(x,0) = f(x).$$

Taking the Fourier cosine transform of this equation gives

$$U(x,0) = A = \hat{f}(k)$$

where $\hat{f}(k)$ is the Fourier cosine transform of $f(x)$. Thus

$$U = \hat{f}(k)e^{-k^2 t}.$$

Now that we've found U we have to invert to get u:

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(k)\cos(kx)e^{-k^2t}\mathrm{d}k.$$

For example if we take

$$f(x) = H(x-a)$$

where H is the heaviside function, we find

$$\hat{f}(x) = \sqrt{\frac{2}{\pi}} \int_0^a \cos(kx)\mathrm{d}x \quad = \quad \sqrt{\frac{2}{\pi}} \left[\frac{\sin(kx)}{k}\right]_0^a = \sqrt{\frac{2}{\pi}} \frac{\sin(ka)}{k}.$$

The solution is this case is

$$u(x,t) = \frac{2}{\pi} \int_0^\infty e^{-k^2t}\, k^{-1} \sin(ka)\cos(kx)\mathrm{d}k.$$

It has to be admitted that this solution does not provide much insight into the form of the solution. But it is relativly simple to calculate the integral numerically for a range of values of t and plot solutions.

2.5 The convolution product

Like the Laplace transform, the Fourier transform also has a convolution product. If $f(x)$ and $g(x)$ are functions of x we define there convolution product by the formula

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(u)g(x-x)\mathrm{d}u.$$

We can show that

$$\mathcal{F}\,[f*g] = \mathcal{F}\,[f]\ \mathcal{F}\,[g].$$

By definition,

$$\mathcal{F}\,[f*g] = \frac{1}{2\pi} \int_{-\infty}^\infty e^{\mathrm{i}kx} \left[\int_{-\infty}^\infty f(u)g(x-u)\mathrm{d}u\right] \mathrm{d}x.$$

Since $e^{\mathrm{i}kx}$ does not depend of u this term can be placed in the secont integral to give

$$\mathcal{F}\,[f*g] = \frac{1}{2\pi} \int_{-\infty}^\infty e^{\mathrm{i}kx} \left[\int_{-\infty}^\infty e^{\mathrm{i}kx} f(u)g(x-u)\mathrm{d}u\right] \mathrm{d}x.$$

We now change the order of integration to give

$$\mathcal{F}\,[f*g] = \frac{1}{2\pi} \int_{u=-\infty}^\infty \int_{x=-\infty}^\infty e^{\mathrm{i}kx} f(u)g(x-u)\mathrm{d}u\mathrm{d}x.$$

Now the term $f(u)$ can be placed in the first integral to give

$$\mathcal{F}[f * g] = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} f(u) \int_{x=-\infty}^{\infty} e^{\mathrm{i}kx} g(x-u)\,\mathrm{d}u\,\mathrm{d}x.$$

Now make a change of variable, setting

$$v = x - u, \qquad \mathrm{d}v = \mathrm{d}x.$$

Then

$$\mathcal{F}[f * g] = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} f(u) \int_{x=-\infty}^{\infty} e^{\mathrm{i}k(u+v} g(v)\,\mathrm{d}v\,\mathrm{d}u.$$

Now splitting the exponential and $2/\pi$ terms we can write

$$\mathcal{F}[f{*}g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{\mathrm{i}ku}\mathrm{d}u \times \mathcal{F}[f{*}g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(v) e^{\mathrm{i}kv}\mathrm{d}v \times = \mathcal{F}[f(x)]\mathcal{F}[g(x)].$$

This completes the proof.

Example: To illustrate the use of the convolution product we consider the prblem of heat diffusion in an infinitely-long rod. The temperature $u(x,t)$ at position x satisfies

$$u_t = u_{xx} \tag{2.25}$$

where again we have taken the thermal diffusivity $\kappa = 1$. Let

$$\mathcal{F}\, u(x,t) = U(k,t).$$

Taking the Fourier transform of (2.25) gives,

$$U_t = -k^2 U, \qquad U = Ae^{-k^2 t},$$

where the arbitrary constant A is determined by the initial condition

$$u(x,0) = f(x), \qquad \text{which implies} U(x,0 = \hat{f}(k),$$

where $\hat{f}(k)$ is the Fourier transform of the initial temperature distribution $f(x)$. Thus

$$U = \hat{f}(k) e^{-k^2 t}.$$

We could write $u(x,t)$ in terms of a convolution, provided we can find $\mathcal{F}^{-1}[e^{-k^2 t}]$. But of course we can use the Fourier inversion formula, which gives

$$\mathcal{F}^{-1}[e^{-k^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t} e^{-\mathrm{i}kx} \mathrm{d}k.$$

The first step in evaluating this integral is to combine the two exponential terms;

$$\mathcal{F}^{-1}[e^{-k^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t + \mathrm{i}kx} \mathrm{d}k.$$

Now we complete the square on the exponent of the exponential term, writing

$$k^2 t + \mathrm{i}kx = t\left[k^2 + \frac{\mathrm{i}kx}{t}\right] = t\left[\left(k + \frac{\mathrm{i}x}{2t}\right)^2 + \frac{x^2}{4t^2}\right].$$

The integral can now be re-written as

$$\mathcal{F}^{-1}\left[e^{-k^2 t}\right] = \frac{1}{\sqrt{2\pi}} e^{-x^2/4t} \int_{-\infty}^{\infty} exp[-t\left(k + \frac{\mathrm{i}x}{2t}\right] \mathrm{d}k.$$

Now make the substitution

$$s = \sqrt{t}\left(k + \frac{\mathrm{i}x}{2t}\right)$$

so that

$$\mathrm{d}s = \sqrt{t}\mathrm{d}k.$$

We then find

$$\mathcal{F}^{-1}\left[e^{-k^2 t}\right] = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/4t) \int_{-\infty}^{\infty} e^{-s^2} \mathrm{d}s.$$

Since the integral term $= \sqrt{\pi}$ we finally obtain

$$\mathcal{F}^{-1}\left[e^{-k^2 t}\right] = \frac{1}{\sqrt{2t}} \exp(-x^2/4t).$$

Now using the convolution product we can write the solution in the form

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-u) \exp(-u^2/4t) \mathrm{d}u.$$

3 Contour integration methods

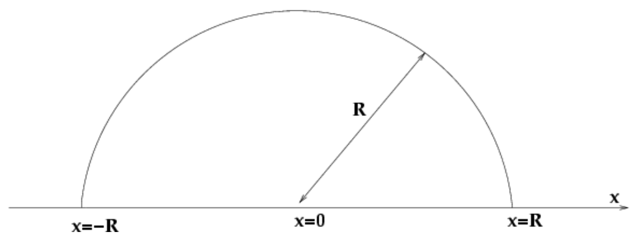
3.1 Jordan's lemma

Sometimes it may be difficult to find simple expressions for the integrals which define a Fourier transform or give its inverse. Contour integration can then be useful. Jordan's lemma shows a wide class of Fourier integrals can be evaluated.

We consider an integral of the form,

$$\int_{-\infty}^{\infty} g(x) e^{\mathrm{i}kx} \mathrm{d}x,$$

where k is a positive constant. We integrate along the closed contour consisting of the segment $[-R \le x \le R]$ of the real axis, and a semicircle of radius R in the upper half of the complex plane, with centre at $x = 0$.



The integral I_1 say, along the semi-circle is given by

$$I_1 = \int_0^{\pi} \mathrm{i}\, \exp\left[\mathrm{i}k(R\cos\theta + \mathrm{i}R\sin\theta)\right] g(Re^{\mathrm{i}\theta})\, Re^{\mathrm{i}\theta}\mathrm{d}\theta.$$

The absolute value of the exponential term is

$$e^{-kR\sin\theta}$$

and we assume that

$$|g(Re^{\mathrm{i}\theta}| \le G(R) \quad \text{say}.$$

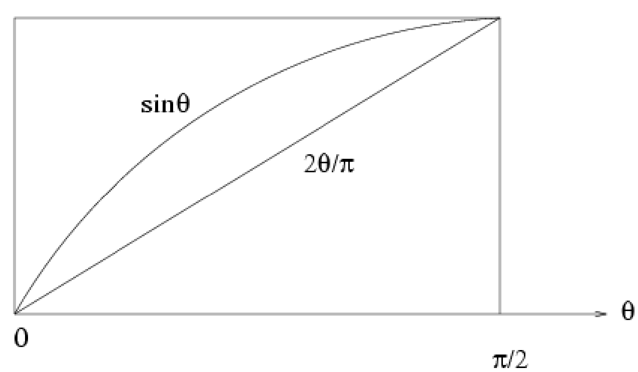
It follows that

$$|I_1| \le 2\int_0^{\pi/2} e^{-R\sin\theta} RG(R)\mathrm{d}\theta.$$

The trick now is use the inequality,

$$\sin\theta \ge \frac{2\theta}{\pi}, \quad 0 \le \theta \le \tfrac{1}{2}\pi,$$

which can be established by the diagram below.



The straight line is the graph of $2\theta/\pi$, and the curve is the graph of $\sin\theta$. Note that

$$\frac{\mathrm{d}^2 \sin(\theta)}{\mathrm{d}\theta^2} = -\sin(\theta,$$

so the graph of $\sin\theta$ has negative curvature and must always lie above the graph of $\sin\theta$. It now follows that

$$|I_1| \leq 2\int_0^{\pi/2} RG(R)e^{-2kR\theta/\pi}\mathrm{d}\theta$$

$$= \frac{\pi G(R)}{k}(1 - e^{-kR}).$$

Provided $G(R)$ is bounded the integral $I_1 \to 0$ as $R \to \infty$, so by Cauchy's theorem,

$$\int_{-\infty}^{\infty} g(x)^{\mathrm{i}kx}\mathrm{d}x = 2\pi\mathrm{i} \times \sum \text{ residues of } g(x)^{\mathrm{i}kx}$$

in the upper half plane.

As an example, evaluate

$$I_2 = \int_{\infty}^{\infty} \frac{e^{\mathrm{i}kx}}{1+x^2}.$$

Clearly $g(x)$ satisfies the conditions of of Jordan's lemma — in fact we can take $G(R) = 1$. The function

$$g(z) = \frac{1}{(z-\mathrm{i})(z+\mathrm{i})}$$

and the only pole we need to consider is at $z = i$ where residue is

$$\frac{1}{2\mathrm{i}}e^{-k}.$$

Thus

$$I_2 = \frac{1}{\pi}e^{-k}.$$

3.2 Inverting the Laplace transform

Up until now, we have relied on ad-hoc methods for inverting the Laplace transform, but a general technique can be derived, making use of the Fourier transform formula.

Let $g(t)$ be some function of t. In order to take a Fourier transform of of $g(t)$ we must be sure it is integrable — i.e. that

$$\int_0^\infty g(t)\mathrm{d}t$$

has a finite value. For this to be the case, $g(t)$ must decrease to zero sufficiently rapidly as $t \to \infty$ To ensure that this is the case we multiply $g(t)$ by the exponential function $e^{-\gamma t}$ where γ is a large enough positive constant. Of course this trick will not work for all functions — for example

$$e^{-\gamma t} e^{t^2}$$

will never tend to zero, no matter how large a value we assign to γ.

With the Laplace we are interested only in positive t, so we chop off the unwanted negative stuff by considering the function

$$h(t) = e^{-\gamma t} H(t).$$

Assuming we have chosen a large enough value of γ we can apply the Fourier integral theorem to $h(t)$ and write

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}kt} \left[\int_{-\infty}^{\infty} e^{\mathrm{i}k\xi} h(\xi) \mathrm{d}\xi \right] \mathrm{d}k.$$

From the definition of $h(t)$ it now follows that

$$e^{-\gamma t} g(t) H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}kt} \left[\int_{-\infty}^{\infty} e^{\mathrm{i}k\xi}\, e^{-\gamma\xi}\, g(\xi) H(\xi) \mathrm{d}\xi \right] \mathrm{d}k.$$

The Heaviside function in the integrand effectively truncates the range of integration to $[0\,,\,\infty]$, and combining the exponential terms,

$$e^{-\gamma t} g(t) H(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}kt} \left[\int_{0}^{\infty} e^{-(\gamma - \mathrm{i}k)\xi}\ g(\xi)\, \mathrm{d}\xi \right] \mathrm{d}k.$$

Now we make a variable change in the k-integration, writing

$$p = \gamma - \mathrm{i}k, \quad , -\mathrm{i}k = p - \gamma, \quad \mathrm{d}k = \mathrm{i}\mathrm{d}p.$$

Then

$$e^{-\gamma t} g(t) = \frac{1}{2\pi} \int_{\gamma + \mathrm{i}\infty}^{\gamma - \mathrm{i}\infty} e^{pt} e^{-\gamma t} \left[\int_0^\infty e^{-p\xi} g(\xi) \mathrm{d}\xi \right] \mathrm{i}\, \mathrm{dp}.$$

Now we cancel the term $e^{-\gamma t}$ on each side, convert the i into a $-1/\mathrm{i}$ and use the minus sign to interchange the limits of the p integration.This produces:

$$g(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{pt} \left[\int_0^\infty e^{-p\xi} g(\xi) \mathrm{d}\xi \right] \mathrm{dp}.$$

The term in square brackets is just the Laplace transform $G(p)$ of $g(t)$ so the final result is

$$g(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{pt} F(p) dp. \tag{3.1}$$

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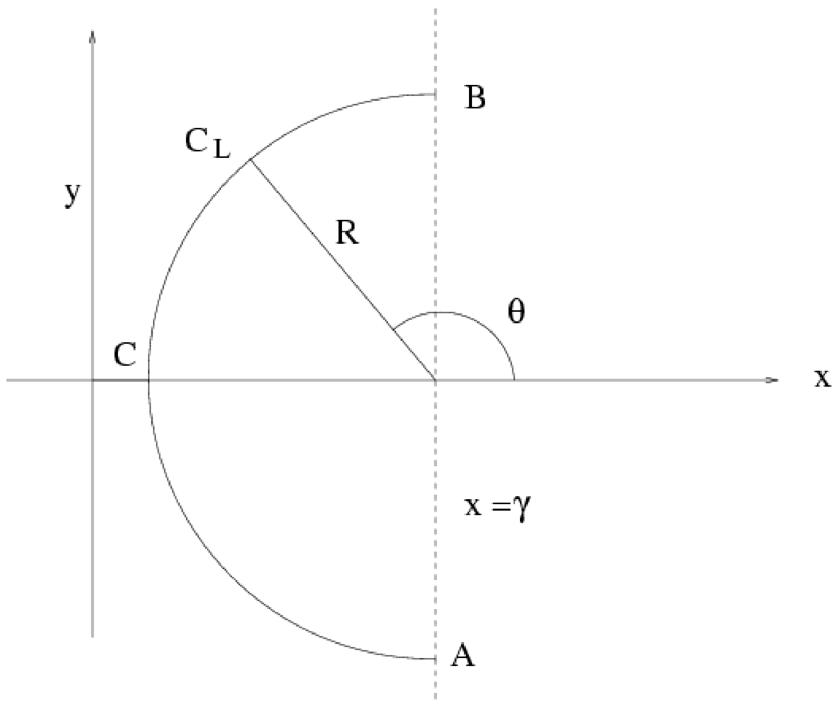
3.3 Laplace transform inversion by contour integration



The formula (3.1) looks difficult to evaluate, but contour integration provides a feasible method. The dashed line in the diagram represents the

$$[\gamma - \mathrm{i}\infty \, , \, \gamma + \mathrm{i}\infty]$$

path given by (3.1), which we close on the left by a semi-circular arc C_L. If we can show for a function $F(p)$ that

$$\lim_{R \to \infty} \int_{C_L} F(p) e^{pt} = 0, \tag{3.2}$$

then the integral along the dashed line will be equal to the integral around the closed contour $ABCA$ in the limit as $R \to \infty$. Then the relatively simple methods of contour integration can be brought into action.

To prove (3.2) we can re-use Jordan's lemma. This shows that the integral of the function $g(x)e^{\mathrm{i}kt}$ around a semi-circular arc of radius R is given by the formula

$$\int_0^{\pi} \exp\big(\mathrm{i}k[R\cos\theta + \mathrm{i}R\sin\theta)]\big)\, g(Re^{\mathrm{i}\theta})\, Re^{\mathrm{i}\theta}\mathrm{d}\theta. \tag{3.3}$$

. There are two main differences — the orientation of the contour and the fact that the function we are integrating is $F(p)e^{pt}$. The difference in orientation means that the angle θ varies from $\pi/2$ to $3\pi/2$, so

$$\int_{BCA} F(p)e^{pt} = \int_{\pi/2}^{3\pi/2} \exp\big(pt[R\cos\theta + \mathrm{i}R\sin\theta)]\big)\, g(Re^{\mathrm{i}\theta})\, Re^{\mathrm{i}\theta}\mathrm{d}\theta.$$

Making the substitution $\phi = \theta - \pi/2$ we find

$$\int_{BCA} F(p)e^{pt} = \int_0^{3\pi} \exp\big(\mathrm{i}pt[R\cos\phi + \mathrm{i}R\sin\phi)]\big)\, g(-\mathrm{i}Re^{\mathrm{i}\phi})\, (-\mathrm{i})Re^{\mathrm{i}\phi}\mathrm{d}\phi.$$

This integral (apart from a constant term) is identical to (3.3), and therefore tends to zero by the same argument, under of course the same conditions. It follows that in the limit as $R \to \infty$, Cauchy's theorem shows that

$$g(t) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{pt}F(p)dp = \frac{1}{2\pi\mathrm{i}} 2\pi\mathrm{i} \times \sum \text{ residues of } F(p)e^{pt}.$$

Simply then,

$$g(t) = \sum \text{ residues of } F(p)e^{pt}. \tag{3.4}$$

Note that γ must be large enough to ensure that the contour encloses all poles of $F(p)e^{pt}$.

Example 1: Find the inverse Laplace transform of

$$F(p) = \frac{1}{p-a}.$$

The only pole of

$$\frac{e^{pt}}{p-a} \quad \text{is at} \quad p = a, \quad \text{with residue} \quad e^{at}.$$

Thus $\mathcal{L}^{-1}[1/(p-a)] = e^{at}$.

Example 2: Find the inverse Laplace transform of

$$F(p) = \frac{1}{p^2+a^2}.$$

We can write
$$\frac{1}{p^2+a^2} = \frac{1}{p^2-(\mathrm{i}a)^2} = \frac{1}{(p-\mathrm{i}a)(p+\mathrm{i}a)}.$$
This time there are poles at $p = \mathrm{i}a$ and at $p = -\mathrm{i}a$. The residue at $p = \mathrm{i}a$ is
$$\frac{e^{\mathrm{i}at}}{2\mathrm{i}a}$$
and at $p = -\mathrm{i}$,
$$\frac{e^{-\mathrm{i}at}}{-2\mathrm{i}a}.$$
Summing the residues we find
$$f(t) = \frac{e^{\mathrm{i}at} - e^{-\mathrm{i}at}}{2\mathrm{i}a} = \frac{2\mathrm{i}\sin(at)}{2\mathrm{i}a} = \frac{1}{a}\sin(at).$$

These two examples have involved only simple poles. More generally we may need to find residues at multiple poles, and the general technique is as follows. Suppose we want to find the residue at $p = a$ of the function
$$\frac{f(p)}{(p-a)^n}. \tag{3.5}$$
We expand $f(p)$ as a Taylor series about the point $p = a$ so the expression (3.5) becomes,
$$\frac{f(a) + f'(a)(p-a) + \frac{1}{2}f''(a)(p-a)^2 \ldots + \ldots (1/n!)f^{(n)}(a)(p-a)^n}{(p-a)^n}.$$
The residue R_a say at $p = a$ is the coefficient of $(p-a)^{-1}$ in this expression — i.e.
$$R_a = \frac{f^{n-1}(a)}{(n-1)!}. \tag{3.6}$$

Example 3: Find the inverse Laplace transform of
$$F(p) = \frac{1}{(p^2+a^2)^2} = \frac{1}{(p-\mathrm{i}a)^2(p+\mathrm{i}a)^2)}.$$
There are double poles at $p = \pm\mathrm{i}a$. Using (3.6) we find that the residue at $p = \mathrm{i}a$ is
$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{e^{pt}}{(p+\mathrm{i}a)^2}\right]_{p=\mathrm{i}a}.$$
$$= \left[\frac{te^{pt}}{(p+\mathrm{i}a)^2} - \frac{2e^{pt}}{(p+\mathrm{i}a)^3}\right]_{p=\mathrm{i}a} = \frac{te^{\mathrm{i}at}}{(2\mathrm{i}a)^2} - \frac{2e^{\mathrm{i}at}}{(2\mathrm{i}a)^3}$$

$$= \frac{te^{\mathrm{i}at}}{-4a^2} - \frac{e^{\mathrm{i}at}}{-4\mathrm{i}a^3} = \frac{ate^{\mathrm{i}at}}{-4a^3} - \frac{\mathrm{i}e^{\mathrm{i}at}}{4a^3}.$$

To calculate the residue at $p = -\mathrm{i}a$ we simply replace p by $\mathrm{i}p$ — i.e. we take the complex conjugate. Thus if we denote our first residue as R the other residue will be R^*, where the star indicates the complex conjugate. The sum of the residues is therefore

$$R + R^* = \quad \text{twice the real part of} \quad R.$$

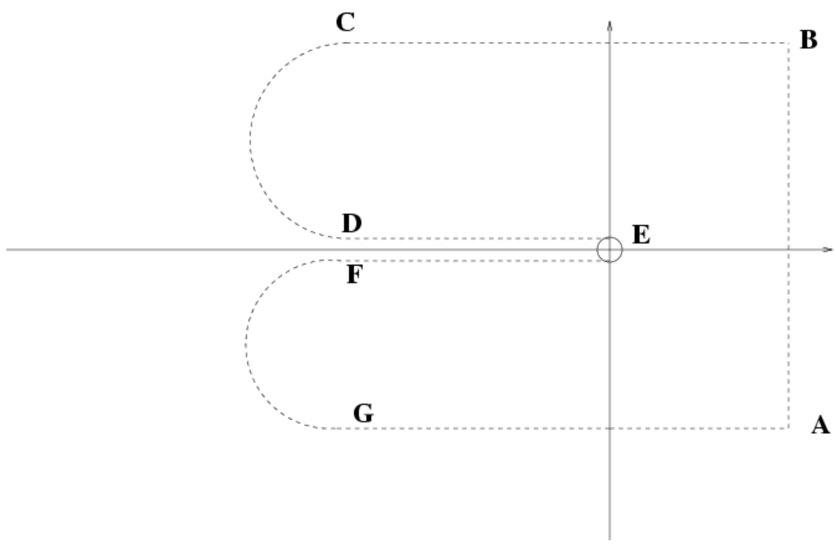
Thus

$$f(t) = \frac{-at\cos(at)}{2a^3} + \frac{\sin(at)}{2a^3} = \frac{1}{2a^3}\big(\sin(at) - at\cos(at)\big).$$

Example 4: Fin the inverse Laplace transform of

$$G(p) = \frac{1}{p}e^{-a\sqrt{p}}.$$

Because of the square root we have to devise an integration contour which avoids the branch point at the origin.



We imagine that contour extends to to $\pm\mathrm{i}\infty$ and to $-\infty$. According to the Laplace transform inversion theorem,

$$g(t) = \frac{1}{2\pi\mathrm{i}}\int_{AB} G(p)e^{pt}\mathrm{d}p, \quad \text{so} \quad \int_{AB} G(p)e^{pt}\mathrm{d}p = 2\pi\mathrm{i}g(t).$$

Since the contour encloses no poles of $G(p)$, Cauchy's theorem shows that

$$2\pi\mathrm{i}\, g(t) + \int_{BCDEFGA} e^{pt}G(p) = 0,$$

and

$$g(t) = -\frac{1}{2\pi \mathrm{i}} \int_{BCDEFGA} G(p) e^{pt}\, G(p)\, \mathrm{d}p. \tag{3.7}$$

Since $|G(p) \to 0$ as $|p| \to \pm\infty$ the contributions along BC and GA vanish. Also the integral along CD and FG vanish a Jordan's lemma type agument. Thus the only integrals we have to consider are the following:

1. The integral along DE
2. The integral along EF
3. The integral around the origin.

.

Contribution 1:

On $DE, p = e^{\mathrm{i}\pi}$ where s is distance from the origin.
Thus $\sqrt{p} = \sqrt{s} e^{\mathrm{i}\pi/2} = -\mathrm{d}s$, and the integral is

$$\int_{\infty}^{0} \frac{e^{-st}}{-s} e^{-\mathrm{i}a\sqrt{s}} (-\mathrm{d}s) = \int_{0}^{\infty} \frac{e^{-st}}{s} e^{-\mathrm{i}a\sqrt{s}} \mathrm{d}s. \tag{3.8}$$

Contribution 2:

On EF $p = se^{-\pi \mathrm{i}} = -s$, $\mathrm{d}p = -\mathrm{d}s$ and $\sqrt{p} = \sqrt{s} e^{-\mathrm{i}\pi/2} = -\mathrm{i}\sqrt{s}$. The contribution from this integral is therefore,

$$\int_{0}^{\infty} \frac{e^{st}}{s} e^{\mathrm{i}a\sqrt{s}} \mathrm{d}s. \tag{3.9}$$

Contribution 3:

On the small circle, as $p \to 0$,

$$\frac{1}{p} e^{-a\sqrt{p}} \to \frac{1}{p},$$

and Cauchy's theorem shows the integral around the small circle is $-2\pi i$. The negative sign appears becuase the integral is taken in the clockwise sense, whereas Cauchy's theorem assumes anticlockwise integration.

Summing contributions (1), (2) and (3) we find

$$\int_{BCDEFGA} G(p) e^{pt}\, G(p)\, \mathrm{d}p = -2\pi \mathrm{i} + \int_{0}^{\infty} e^{-st}\, \frac{2i \sin(a\sqrt{s}\,)}{s}.$$

Finally (3.7) gives

$$g(t) = 1 - \frac{1}{\pi} \int_0^\infty e^{-st} \, \frac{\sin(a\sqrt{s}\,)}{s} \, \mathrm{d}s. \tag{3.10}$$

It can be shown that this integral is

$$\operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right).$$

See page 209 of *Theory and problems of Laplace transforms.*

Appendix 1

Expressions for Laplacian in cylindrical polars

In cylindrical polars $r^2 = x^2 + y^2$, so differentiating implicitly with respect to x we find,

$$2r\frac{\partial r}{\partial x} = 2x, \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{similarly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}. \tag{.11}$$

Also $x = r\cos\theta$ so differentiating implicity with respect to x,

$$1 = \frac{x}{r}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial x}, \qquad y\frac{\partial\theta}{\partial x} = \frac{x^2}{r} - 1.$$

Thus

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2}, \qquad \text{similarly,} \qquad \frac{\partial\theta}{\partial y} = \frac{x}{r^2}. \tag{.12}$$

To find the Laplacian of a function $f(r,\theta)$ we first differentiate with respect to x using the chain rule:

$$f_x = f_r\frac{\partial r}{\partial x} + f_\theta\frac{\partial\theta}{\partial x} = f_r\frac{x}{r} - f_\theta\frac{y}{r^2}.$$

Now, differentiating again with respect to x,

$$f_{xx} = \frac{f_r}{r} + \frac{x^2}{r^2}\left(f_{rr} - \frac{f_r}{r}\right) + \frac{y^2}{r^4}f_{\theta\theta} + \frac{2xy}{r^4}f_\theta - \frac{xyf_{\theta r}}{r^2} \tag{.13}$$

Similary we can show that

$$f_{yy} = \frac{f_r}{r} + \frac{y^2}{r^2}\left(f_{rr} - \frac{f_r}{r}\right) + \frac{x^2}{r^4}f_{\theta\theta} - \frac{2xy}{r^4}f_\theta + \frac{xyf_{\theta r}}{r^2}. \tag{.14}$$

Adding (.13) and (.14) and remembering that $x^2 + y^2 = r^2$ we find that

$$\nabla^2 f = f_{xx} + f_{yy} = f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2}.$$

Appendix 2

Gamma function

The gamma function is defined by setting

$$\Gamma(x) = \int_0^\infty e^{-s} s^{x-1} \mathrm{d}s.$$

Integrating by parts we find

$$\Gamma(x) = \left[\frac{e^{-s} s^x}{x} \right]_0^\infty - \int_0^\infty \frac{-s^x e^{-s}}{x} \mathrm{d}s =$$

$$\frac{1}{x} \int_0^\infty s^x e^{-s} \mathrm{d}s = \frac{1}{x} \Gamma(1+x).$$

This proves the recursion formula for the gamma function,

$$\Gamma(x+1) = x\Gamma(x).$$

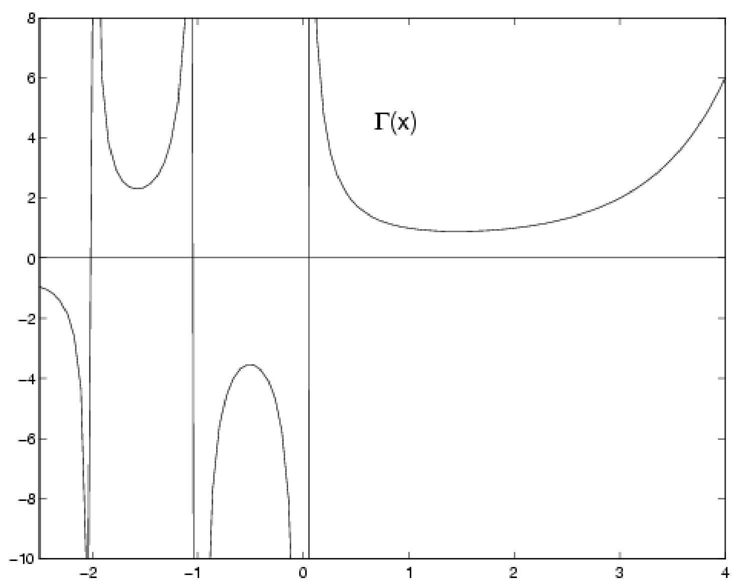
Since $\Gamma(1) = 1$ we have $\Gamma(2) = 1$, $\Gamma(3) = 2$ etc. In general if n is an integer $\Gamma(n) = (n-1)!$

Note also that

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-s} s^{-1/2} \mathrm{d}s.$$

Making the substitution $s = t^2$ gives

$$\Gamma(\tfrac{1}{2}) = 2 \int_0^\infty e^{-t^2} \mathrm{d}t = \sqrt{\pi}.$$



Note that $\Gamma(x)$ is singular at $x = 0$ and at the negative integers.