Gram lines and the average of the real part of the Riemann zeta function

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The contours $\Im\Lambda(s) = 0$ of the function which satisfies $\zeta(1-s) = \Lambda(s)\zeta(s)$ cross the critical strip on lines which are almost horizontal and straight, and cut the critical line alternately at Gram points and points where $\zeta(s)$ is imaginary. The real part of $\zeta(s)$, when averaged in a modified manner, for fixed values of σ over the values on the "Gram lines", satisfies a relation which extends a theorem of Titchmarsh, (namely that the average of $\zeta(s)$ over the Gram points is 2), to the right hand side of the critical strip.

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1. INTRODUCTION

First we set out some standard notation and properties. Let $\Phi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)$, so $\xi(s) = \Phi(s)\zeta(s)$ satisfies $\xi(\frac{1}{2}+s) = \xi(\frac{1}{2}-s)$ (a form of the functional equation), and also $\xi(\frac{1}{2}+it)$ is real for $t \in \mathbb{R}$.

Write

$$\xi(\tfrac{1}{2}+it) = -f(t)e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it)$$

where $f(t)$ is real and positive, and then define $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. So $Z(t)$ is real when t is real and its zeros in $\mathbb{R}$ correspond to the zeros of $\zeta(s)$ on $\sigma = \Re s = \frac{1}{2}$, the critical line.

Define Gram points $(\frac{1}{2}+ig_n)$ as points on the critical line which satisfy $\vartheta(g_n) = n\pi$ for integral $n \geq -1$. Then at these points $\zeta(s)$ is real,

Now let $\Lambda(s) = 2(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi s}{2})$ so $\zeta(1-s) = \Lambda(s)\zeta(s)$ (another form of the functional equation). By applying the property $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ of the gamma function it follows that $\Lambda(s)\Lambda(1-s) = 1$ and hence that $\Lambda(s) = \Phi(1-s)/\Phi(s)$.

Now we summarize the content of this paper regarding the Gram lines and average values of the real part of $\zeta(s)$ on (vertical lines through) these lines. The origin of the ideas underlying these results was the observation of the phase portrait of $\dot{s} = \Lambda(s)$. See Fig.1 and the article [2] for background material on this approach.

The contours $\Im\Lambda(s) = 0$ of the function cross the critical strip on lines which are almost horizontal and straight and cut the critical line at (1) the Gram points (so are called simply "Gram lines") or (2) the points where $\zeta(s)$ is imaginary. The contours are shown to satisfy the equations, for $n \in \mathbb{Z}$,

$$\frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} - \frac{(\sigma-\frac{1}{2})^2}{4t} + O(\frac{1}{t^3}) = \frac{n\pi}{2}.$$

They differ from horizontal lines by $O(1/t)$. On these contours $\Re\Lambda(s)$ is never zero and is strictly monotonic with

$$\Re\Lambda(s) = (-1)^n\big(\frac{t}{2\pi}\big)^{\sigma-\frac{1}{2}}\{1 + O(\frac{1}{t^2})\}.$$

This monotonicity is then applied to zeta to show that the value of the modulus of the derivative, at simple zeros at mirror image points (with respect to the critical line), is never the same.

It is then shown that the contours $\Im\Phi(s) = 0$ also cut the critical line at the Gram points but at an angle to the horizontal which is asymptotically $-\pi/(2\log t)$. They differ from intervals at that slope by $O(1/t)$ and have equations given implicitly by:

$$\frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} + \frac{\pi(\sigma-1)}{4} + \frac{1}{12t} - \frac{(\sigma-1)^2}{4t} + O(\frac{1}{t^3}) = n\pi.$$

The main result of the paper is then given as a corollary to Theorem 3.1. A theorem of Titchmarsh [3], gives 2 as the average of the real part of $\zeta(\frac{1}{2} + ig_n)$ i.e.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le n\le N}[\Re\zeta(\tfrac{1}{2} + ig_n) - 2] = 0.$$

When the real part of $\zeta(\sigma + ig_n)$ is modified and then averaged, for fixed values of σ, over its values, it satisfies a relation which extends this theorem. If $\frac{1}{2} \le \sigma < 1$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le n\le N}[\Re\zeta(\sigma + ig_n) - 1 - (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma}] = 0.$$

The proof of Theorem 3.1 shows that the argument to $\zeta(s)$ used for these averages may be chosen either on the Gram lines through g_n or on the corresponding horizontal interval. We have chosen the latter for simplicity.

The numerical evidence for this theorem is very strong, with better convergence than that obtained analytically in Corollary 3.1. See Fig. 3. Also, this numerical evidence indicates that the average result should apply across the whole of the critical strip and beyond, but we have been unable to prove this.

2. CONTOURS WHERE $\Lambda(S)$ AND $\Phi(S)$ ARE REAL

Fig. 2 shows a few contours where $\Im\Lambda(s) = 0$.

Lemma 2.1. *(a) The contours* $\Im\Lambda(s) = 0$ *cut across the critical strip symmetrically.*

(b) They cut the critical line at the points $s = \frac{1}{2} + i\gamma$ *where* $\zeta(s)$ *is real (Gram points) or imaginary, with value* $\Lambda(s) = \pm 1$*. If* $\Lambda(s) = 1$ *then* $\zeta(s)$ *is real and if* $\zeta(s)$ *is a simple zero it is a center. If* $\Lambda(s) = -1$ *then* $\zeta(s)$ *is imaginary and if* $\zeta(s)$ *is a simple zero it is a node.*

(c) If $\Im(s) \neq 0$ *then* $\Lambda(s) \neq 0$.

Proof. (a) For real $x, t > 0$, let $P_+ = \frac{1}{2} + x + it$ and $P_- = \frac{1}{2} - x + it$. Because $\Lambda(P_+)\Lambda(\overline{P_-}) = 1$, if $\Lambda(P_+)$ is real, so is $\Lambda(P_-)$. Therefore the contours are symmetric about $\sigma = \frac{1}{2}$.

(b) If $s = \frac{1}{2} + it$ and $\Im\Lambda(s) = 0$ then $\Lambda(s)$ and $\Lambda(\overline{s})$ are real. Therefore $\Lambda(\frac{1}{2} + it)\Lambda(\frac{1}{2} - it) = 1$ implies $\Lambda(\frac{1}{2} + it)^2 = 1$ so $\Lambda(\frac{1}{2} + it) = \pm 1$. If $\Lambda(\frac{1}{2} + it) = 1$, by the functional equation $\zeta(1 - s) = \Lambda(s)\zeta(s)$, $\overline{\zeta(\frac{1}{2} + it)} = \zeta(\frac{1}{2} - it) = \zeta(\frac{1}{2} + it)$ so $\zeta(\frac{1}{2} + it)$ is real. If $\Lambda(\frac{1}{2} + it) = -1$ then $\overline{\zeta(\frac{1}{2} + it)} + \zeta(\frac{1}{2} + it) = \zeta(\frac{1}{2} - it) + \zeta(\frac{1}{2} + it) = -\zeta(\frac{1}{2} + it) + \zeta(\frac{1}{2} + it) = 0$ so $\zeta(\frac{1}{2} + it)$ is pure imaginary.

If $P = \frac{1}{2} + it$ is a simple zero of $\zeta(s)$ and $\Lambda(P) = 1$, then differentiating the functional equation gives $-\zeta'(\overline{P}) = \zeta'(P)$ so $\zeta'(P)$ is imaginary and the zero is a center. If $\Lambda(P) = -1$, $\zeta'(P)$ would be real, so the zero is a node.

(c) Because $\Gamma(s)$ is never zero and the zeros of $\cos(\pi s/2)$ are on the real axis, if $\Im(s) \neq 0$ then $\Lambda(s) \neq 0$. ∎

Lemma 2.2. *The contours of* $\Im\Lambda(s) = 0$*, for* $0 \leq \sigma \leq 1$*, differ from intervals parallel to the x-axis through those points by* $O(1/t)$*. Indeed, the contours satisfy the equations, for* $n \in \mathbb{Z}$*,*

$$\frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} - \frac{(\sigma - \frac{1}{2})^2}{4t} + O(\frac{1}{t^3}) = \frac{n\pi}{2} \qquad \textit{(1),}$$

where n *is even for lines through Gram points and* n *is odd for lines through the points where* $\zeta(s)$ *is imaginary.*

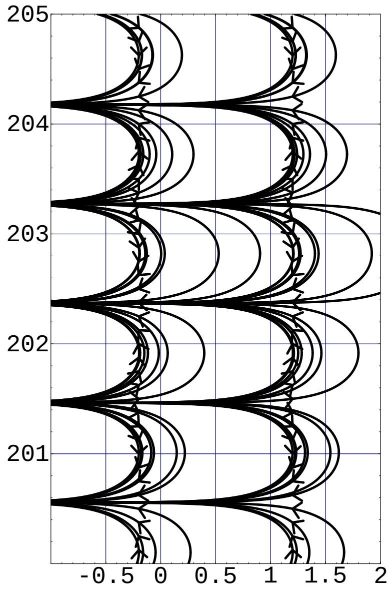

FIG. 1. Phase portrait for $\dot{s} = \Lambda(s)$ with $-1 \leq \sigma \leq 2$ and $200 \leq t \leq 205$.

Proof. (Because the manipulation is somewhat lengthy, but the steps easy to describe, we simply describe those steps taken to obtain the given equation.) Let $s = \sigma + it$ with $0 \leq \sigma \leq 1$. Write down the equation $\Lambda(s) = \Lambda(\overline{s})$ and expand it using the asymptotic Stirling series approximation for the gamma function

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \frac{1}{12s} + O(\frac{1}{t^3}),$$

retaining terms of order $O(1/t)$. The $O(1/t)$ bound follows by implicit differentiation and the mean value theorem. ▌

Note that

$$\vartheta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O(\frac{1}{t^5})$$

satisfies $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it)$ which is real when t is real. Note also that the expression in the lemma above gives the correct value for Gram points at $\sigma = \frac{1}{2}$.

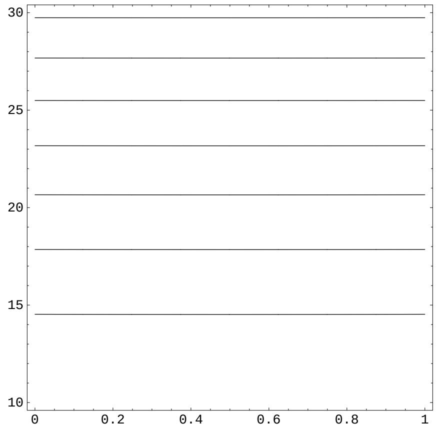

FIG. 2. Some contours for $\Im\Lambda(s)=0$ with $0\le t\le 30$ and $0\le\sigma\le 1$.

LEMMA 2.3. *For* $t=\Im s$ *sufficiently large, on the contours* $\Im\Lambda(s)=0$, $\Re\Lambda(s)$ *is strictly monotonic with*

$$\Re\Lambda(s)=(-1)^n\big(\frac{t}{2\pi}\big)^{\sigma-\frac{1}{2}}\{1+O(\frac{1}{t^2})\}.$$

Proof. Consider the equation $2\Re\Lambda(s)=\Lambda(s)+\Lambda(\overline{s})$ where $s=\sigma+it$ and $t>0$ is sufficiently large. Expand the trigonometric terms as exponentials and set those with $-t$ in the exponent to zero. Then use the Stirling series approximation for $\log\Gamma(s)$ to obtain the expression

$$\begin{aligned}\Re\Lambda(s) &= \exp\big((\sigma-\frac{1}{2})\log(\frac{t}{2\pi})+O(\frac{1}{t^2})\big)\times\\ &\quad \cos\big(2[\frac{t}{2}\log\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}-\frac{(\sigma-\frac{1}{2})^2}{4t}+O(\frac{1}{t^3})]\big)\\ &= \exp(f(\sigma,t))\times\cos(2\vartheta_\sigma(t)) \text{ say.}\end{aligned}$$

Then

$$\frac{\partial f}{\partial\sigma}=\frac{1}{2}\log(\frac{t}{2\pi})+O(\frac{1}{t^2})$$

which is positive for t sufficiently large. On the contours, the argument of the cosine term is $n\pi+O(1/t)$ so is asymptotically ±1 uniformly in σ, giving the strict monotonicity of $\Re\Lambda(s)$. ▌

Remark: A similar calculation to that used in the previous lemma leads, for $\sigma \in \mathbb{C}$ with $\Re s \neq 0$, to

$$\Im\Lambda(s) = (\frac{t}{2\pi})^{\sigma-\frac{1}{2}}\{1+O(\frac{1}{t^2})\}\sin(2\vartheta_\sigma(t)).$$

For $x, y > 0$ let $P_+ := \frac{1}{2}+x+iy,\ P_- := \frac{1}{2}-x+iy.$

Theorem 2.1. *Let $\zeta(s)$ have a zero of order m at $s = P_+$ for some $x, y > 0$. Then $|\zeta^{(m)}(P_-)| > |\zeta^{(m)}(P_+)|$. Hence, if $m = 1$, $\zeta'(P_+) \neq \pm\zeta'(P_-)$.*

Proof. Differentiating $\zeta(1-s) = \Lambda(s)\zeta(s)$ m times leads to

$$(-1)^m\zeta^{(m)}(\overline{P_-}) = \Lambda(P_+)\zeta^{(m)}(P_+).$$

Because $|\Lambda(s)| = \left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}}[1+O(\frac{1}{t^2})]$, considering the explicit value of the constant shows that $|\Lambda(P_+)| > 1$ if $t > 10$. ▌

Lemma 2.4.

$$\arg\Phi(P_+) - \arg\Phi(P_-) = \frac{\pi x}{2} + O(\frac{1}{y}).$$

Proof. First note that

$$\arg\Phi(P_+) - \arg\Phi(P_-) = \Im\log\Gamma(\frac{1}{4}+\frac{x}{2}+\frac{iy}{2}) - \Im\log\Gamma(\frac{1}{4}-\frac{x}{2}+\frac{iy}{2}).$$

The result follows by expanding and simplifying this expression in the normal manner. ▌

Lemma 2.5. *The contours $\Im\Phi(s) = 0$ cut the critical line at the Gram points at an angle to the horizontal which is asymptotically $-\pi/(2\log t)$. They differ from intervals at that slope by $O(1/t)$ and have equations given implicitly by:*

$$\frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} + \frac{\pi(\sigma-1)}{4} + \frac{1}{12t} - \frac{(\sigma-1)^2}{4t} + O(\frac{1}{t^3}) = n\pi.$$

Proof. Let $s = \frac{1}{2}+it$. Then, since $\Phi(s)$ is real, $\overline{\Phi(\frac{1}{2}+it)} = \Phi(\frac{1}{2}-it) = \Phi(\frac{1}{2}+it)$. Thus, by Lemma 1.2,

$$\frac{\Phi(\frac{1}{2}-it)}{\Phi(\frac{1}{2}+it)} = 1 = \Lambda(\frac{1}{2}+it).$$

Again, by the remark at the end of paragraph 4 of the introduction and Lemma 2.1, this shows that $\frac{1}{2} + it$ is a Gram point.

The remainder of the derivation is similar to that of Lemma 2.2 so is omitted. ▌

3. AVERAGES

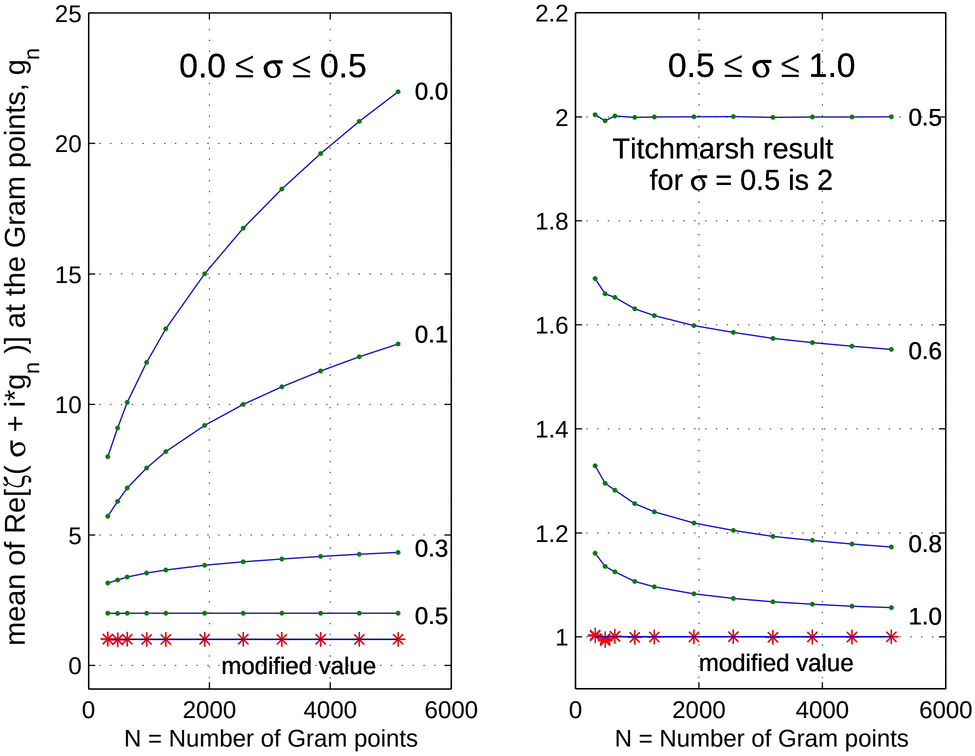

FIG. 3. Averages of $\Re\zeta(s)$ on each side of the critical strip for some values of σ. Modified values are averages of $\Re\zeta(\sigma + ig_n) - (g_n/(2\pi))^{\frac{1}{2}-\sigma}$, and by Theorem 3.1 should tend to 1. Using 5120 Gram points resulted in agreement with the theorem to within 1 in 10^4.

Lemma 3.1. *Let $1 \leq M < N$ and let $n \geq 2$. Then*

$$\sum_{\nu=M+1}^{N} \cos(t_\nu \log n) = O(N^{\frac{1}{2}} \log N)$$

where the implied constant is absolute.

Proof. We begin by following the argument set out in [4, Section 10.6]. Let, for all $\nu \in \mathbb{R}$ with $\nu > 0$, $\vartheta(t_\nu) = \nu\pi$ and define the function ϕ by

$$\phi(\nu) := \frac{t_\nu \log n}{2\pi}.$$

Then

$$\phi'(\nu) = \frac{\log n}{2\vartheta'(t_\nu)},$$

and therefore

$$\phi''(\nu) = -\pi \log n \frac{\vartheta''(t_\nu)}{2\vartheta'(t_\nu)^3}.$$

Note that

$$\begin{aligned} \vartheta(t) &= \frac{1}{2} t \log t + O(t), \quad \vartheta'(t) = \frac{1}{2} \log t + O(1), \\ \vartheta''(t) &= \frac{1}{2t} + O(\frac{1}{t^2}), \text{ and } t_\nu \sim \frac{2\pi\nu}{\log \nu}. \end{aligned}$$

Therefore

$$-\phi''(\nu) = \frac{2\pi \log n}{t_\nu \log^3 t_\nu} + O(\frac{1}{t_\nu^2}).$$

Hence if $L := 2\pi \log n / t_N \log^3 t_N$ and M is sufficiently large (depending on absolute constants), for all ν with $M < \nu \leq N$, $-\phi''(\nu) \geq L > 0$.

Now we depart from [4], applying [1, Corollary 8.12] with $f = -\phi$. This gives directly

$$\sum_{\nu=M+1}^{N} \cos(t_\nu \log n) = O(t_N^{\frac{1}{2}} \log t_N) = O(N^{\frac{1}{2}} \log N).$$

Finally note that the implied constant depends only on $1/\log n$ so can be made absolute, and the proof of the lemma is complete. ▌

The following theorem is illustrated in Fig. 3.

THEOREM 3.1. *Let $\frac{1}{2} \leq \sigma < 1$ and let N be whole number. Then, for all $\epsilon > 0$ and $N \to \infty$:*

$$\sum_{1 \leq n \leq N} [\Re\zeta(\sigma + ig_n) - 1 - (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma}] = O(N^{(1+\sigma)/2+\epsilon}),$$

where the implied constant depends on σ and ϵ.

Proof. (1) Begin with the approximate functional equation [4, page 79, Eqn. 4.12.4] in the form

$$\zeta(s) = \sum_{1 \leq n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} + \quad \chi(s) \sum_{1 \leq n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}} + O(\frac{1}{t^{\sigma/2}})$$

where $t > 0$ and $s = \sigma + it$. Note that $\chi(s) = 1/\Lambda(s)$, so, using Lemma 2.3

$$\begin{aligned}
\cos(2\vartheta_\sigma(g_\nu)) &= 1 + O(\frac{1}{g_\nu^2}), \\
\sin(2\vartheta_\sigma(g_\nu)) &= O(\frac{1}{g_\nu}), \\
\Re\chi(g_\nu) &= (\frac{g_\nu}{2\pi})^{\frac{1}{2}-\sigma}\{1 + O(\frac{1}{g_\nu^2})\}\cos(2\vartheta_\sigma(g_\nu)), \text{ and} \\
\Im\chi(g_\nu) &= (\frac{g_\nu}{2\pi})^{\frac{1}{2}-\sigma}\{1 + O(\frac{1}{g_\nu^2})\}\sin(2\vartheta_\sigma(g_\nu)).
\end{aligned}$$

Now take the real part of the approximate functional equation, and evaluate both sides with $t = g_\nu$, corresponding to an even value of n in equation (1) of Lemma 2.2, to obtain

$$\begin{aligned}
\Re\zeta(\sigma + ig_\nu) &= 1 + (\frac{g_\nu}{2\pi})^{\frac{1}{2}-\sigma}\{1 + O(\frac{1}{g_\nu^2})\}^2 \\
&+ \sum_{2 \leq n \leq \sqrt{\frac{g_\nu}{2\pi}}} \cos(g_\nu \log n)[\frac{1}{n^\sigma} + (\frac{g_\nu}{2\pi})^{\frac{1}{2}-\sigma}\{1 + O(\frac{1}{g_\nu^2})\}^2 \frac{1}{n^{1-\sigma}}] \\
&+ \sum_{2 \leq n \leq \sqrt{\frac{g_\nu}{2\pi}}} \sin(g_\nu \log n)\frac{1}{n^{1-\sigma}}(\frac{g_\nu}{2\pi})^{\frac{1}{2}-\sigma} O(\frac{1}{g_\nu}) \\
&+ O(\frac{1}{g_\nu^{\sigma/2}}).
\end{aligned}$$

Therefore

$$\begin{aligned}
f(\nu,\sigma) &:= \Re\zeta(\sigma+ig_\nu) - 1 - \Big(\frac{g_\nu}{2\pi}\Big)^{\frac{1}{2}-\sigma}\\
&= \sum_{2\le n\le\sqrt{g_\nu/2\pi}} \frac{\cos(g_\nu\log n)}{n^\sigma} + \Big(\frac{g_\nu}{2\pi}\Big)^{\frac{1}{2}-\sigma} \sum_{2\le n\le\sqrt{g_\nu/2\pi}} \frac{\cos(g_\nu\log n)}{n^{1-\sigma}}\\
&+ O\Big(\frac{1}{g_\nu^{\sigma/2}}\Big),
\end{aligned}$$

where the third term and three error terms have been absorbed into one error term.

Call the first term on the right $f_1(\nu,\sigma)$ and the second $f_2(\nu,\sigma)$ so

$$\begin{aligned}
\sum_{M<\nu\le N} f(\nu,\sigma) &= \sum_{M<\nu\le N} f_1(\nu,\sigma) + \sum_{M<\nu\le N} f_2(\nu,\sigma) + O\Big(\sum_{M<\nu\le N} \frac{1}{g_\nu^{\sigma/2}}\Big)\\
&= F_1(\sigma) + F_2(\sigma) + O(N^{1-\sigma/2+\epsilon}).
\end{aligned}$$

(2) Consider the first term on the right and let $\tau := \max(g_{M+1}, 2\pi n^2)$. Then

$$\begin{aligned}
F_1(\sigma) &= \sum_{\nu=M+1}^{N} \sum_{2\le n\le\sqrt{g_\nu/2\pi}} \frac{\cos(g_\nu\log n)}{n^\sigma}\\
&= \sum_{2\le n\le\sqrt{g_N/2\pi}} \frac{1}{n^\sigma} \sum_{\tau\le g_\nu\le g_N} \cos(g_\nu\log n)\\
&= O\big(g_N^{\frac{1-\sigma}{2}+\frac{1}{2}}\log g_N\big) = O(N^{1-\sigma/2+\epsilon}),
\end{aligned}$$

where the implied constant depends on σ and ϵ.

(3) In considering $F_2(\sigma)$ we restrict our attention to $\frac{1}{2}<\sigma<1$. Then, because $(g_\nu/2\pi)^{\frac{1}{2}-\sigma}$ is a positive decreasing sequence, we can apply the method of partial summation, and the previous lemma, to the inner sums

$$\sum_{\tau\le g_\nu\le g_N} \cos(g_\nu\log n)\Big(\frac{g_\nu}{2\pi}\Big)^{\frac{1}{2}-\sigma}$$

to bound these sums by $O(N^{\frac{1}{2}}\log N)$ and so obtain the bound $F_2(\sigma) = O(N^{\frac{1}{2}+\sigma/2+\epsilon})$.

(4) Combining the estimates from (2) and (3) gives the final result for $\frac{1}{2}<\sigma<1$. Note that we can absorb the neglected first M terms into the sum because the exponent of N is positive in the given range.

(5) If $\sigma = \frac{1}{2}$ the result is that of Titchmarsh [3, 4]. ▮

Numerical evidence strongly suggests this result extends to the whole of the critical strip and beyond. See Fig.3. Indeed we conjecture that this is so.

COROLLARY 3.1. *Let* $\frac{1}{2} \leq \sigma < 1$. *Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} [\Re\zeta(\sigma + ig_n) - 1 - (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma}] = 0.$$

COROLLARY 3.2. *For all* σ *with* $0 < \sigma < 1$ *and* ϵ *with* $0 < \epsilon < 1$ *the inequality*

$$\sum_{1\leq n\leq N} \Re\zeta(\sigma + ig_n) \geq (1-\epsilon)N$$

holds for all N *sufficiently large.*

Proof. (1) Let $\frac{1}{2} \leq \sigma < 1$. By the previous theorem, there exists a constant c_1 such that for all N sufficiently large

$$-c_1 N^{(1+\sigma)/2+\epsilon} \leq \sum_{1\leq n\leq N} \Re\zeta(\sigma+ig_n) - N - \sum_{2\leq n\leq N} (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma} \leq c_1 N^{(1+\sigma)/2+\epsilon}.$$

But for all N sufficiently large

$$\sum_{2\leq n\leq N} (\frac{g_n}{2\pi})^{\frac{1}{2}-\sigma} \geq \sum_{2\leq n\leq N} (\frac{n}{\log n})^{\frac{1}{2}-\sigma-\epsilon/2} \geq \sum_{2\leq n\leq N} n^{\frac{1}{2}-\sigma-\epsilon} \geq c_2 N^{3/2-\sigma-\epsilon}.$$

Therefore

$$\sum_{1\leq n\leq N} \Re\zeta(\sigma + ig_n) \geq N + c_2 N^{3/2-\sigma-\epsilon} - c_1 N^{(1+\sigma)/2+\epsilon} \geq (1-\epsilon)N$$

for some positive constant c_3 and all N sufficiently large (depending on c_3, σ, ϵ).

(2) For $0 < \sigma < \frac{1}{2}$ we need only observe, since $\Re\Lambda(\sigma+ig_\nu)$ is an increasing function of σ which is less than 1 for $0 < \sigma < \frac{1}{2}$, that $\Re\zeta(\sigma + ig_\nu) > \Re\zeta(1-\sigma + ig_\nu)$ so, by part (1) of this proof,

$$\sum_{1\leq n\leq N} \Re\zeta(\sigma + ig_n) \geq (1-\epsilon)N,$$

also and the result follows. ▮

Corollary 3.3. *For each σ with $0 < \sigma < 1$ there exist an infinite number of positive integers n with $\Re\zeta(\sigma + ig_n) > 0$.*

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