

An arithmetic extension of van der Waerden's theorem on arithmetic progressions

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A short proof is given of a new extension of the classical theorem of van der Waerden on arithmetic progressions of integers. The extension is based on Furstenberg's topological dynamics. It shows, for example, that in any finite partition of the non-negative integers, one element contains, for every l , for some m and n , the m 'th, the $m + n$ 'th, and so on up to the $m + ln$ 'th prime number.

Key Words: recurrent point, partition, arithmetic progression

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1. INTRODUCTION

The classical van der Waerden theorem of 1927 states that any finite partition of the integers, contains an element with arithmetic progressions of arbitrary length. The theorem has been proved in a number of ways including combinatorial, topological and categorial, see for example [2, 3, 4, 5, 6, 8]. It has also been extended and generalized, see for example [7, 9]. Names associated with this work include Deuber, Graham, Hales, Jewett, Khinchin, Furstenberg, Cohen, Taylor, Weiss and Witt.

The extension stated and proved in this paper is given in terms of a given arbitrary function on the non-negative integers with non-negative integer values. An element of the partition will contain, for each l and starting value, a set of l iterates of the function values, the number of iterates being the elements of an arithmetic progression.

The extension involves a simple modification of Furstenberg's proof of van der Waerden's theorem: the shift map $n \rightarrow n + 1$ is replaced by the given function. Because this gives rise to a continuous mapping on the space of sequences, Furstenberg's theorem on multiple recurrences is able to be employed.

A few examples of the use of this device are given. Maps such as $n \rightarrow 2n$, and $n \rightarrow 2n + 1$ (or more generally $n \rightarrow \alpha n + \beta$ all yield combinatorial information. If A is any infinite subset of the non-negative integers, then by defining the function to give the next strictly greater element of A , it follows directly that if a_i is the i 'th element of A , then an element of the partition contains $\{a_m, a_{m+n}, a_{m+2n}, \dots, a_{m+tn}\}$. (This work was first presented at the Devonport (Auckland) Topology Fest in February 2002.)

2. VAN DER WAERDEN'S THEOREM EXTENDED

Let $Z = \{0, 1, 2, \dots\}$ be the non-negative integers. Let $r \in \mathbb{N}$ have $r \geq 2$ and let $\Lambda = \Omega^Z$. Define a metric d on Ω by

$$d(x, x') = \inf\left\{\frac{1}{k+1} : x(i) = x'(i), 0 \leq i \leq k\right\}.$$

Then the metric space (Ω, d) is compact in the induced topology.

Let $\theta : Z \rightarrow Z$ be any function. Define a map $T : \Omega \rightarrow \Omega$ by the rule

$$(Tx)(n) = x(\theta(n)).$$

Then for each $m \in \mathbb{N}$, the composite function T^m satisfies $(T^m x)(n) = x(\theta^m(n))$ for all $n \geq 1$.

LEMMA 2.1. *For every $\theta : Z \rightarrow Z$, the function T is continuous.*

Proof. For each $n \geq 0$ let $m_n = \max\{n, \theta(0), \theta(1), \dots, \theta(n)\}$. Then $m_n \rightarrow \infty$ and

$$\theta(\{0, 1, \dots, n\}) \subset \{0, 1, 2, \dots, m_n\}.$$

Hence if $d(x, x') < 1/m_n$ then $d(Tx, Tx') < 1/n$ showing T is continuous on Ω . ■

THEOREM 2.1 (Furstenberg Recurrence, [2, 3]). *Let X be a compact metric space, $T : X \rightarrow X$ a continuous function, and $l \geq 1$ be given. Then there exists an $x \in X$ and subsequence $n_k \rightarrow \infty$ of \mathbb{N} with*

$$T^{n_k} x \rightarrow x, T^{2n_k} x \rightarrow x, \dots, T^{ln_k} x \rightarrow x.$$

THEOREM 2.2. *Let $\theta : Z \rightarrow Z$ be any function and let*

$$Z = \cup_{i=1}^r C_i$$

be a representation of Z as a finite union of subsets. Let $a \in Z$ be given. Then for every $l \in \mathbb{N}$ there is an index j such that there are numbers m and n with

$$\{\theta^m(a), \theta^{m+n}(a), \theta^{m+2n}(a), \dots, \theta^{m+ln}(a)\} \subset C_j.$$

Proof. Use the definitions of Λ, Ω and d given above. Define a sequence $\xi : Z \rightarrow \Lambda$

$$C_i = \{n \in Z : \xi(n) = i\}, 1 \leq i \leq r.$$

Let

$$X = \overline{\{T^m \xi : m = 0, 1, 2, \dots\}}.$$

Then $T : X \rightarrow X$. By Furstenberg Recurrence, there is an $n \in Z$ such that

$$x(a) = T^n x(a) = T^{2n} x(a) = \dots = T^{ln} x(a).$$

But $\{T^m \xi\}$ is dense in X . Therefore there is an m such that $T^m \xi$ and x agree on $[0, L]$ where $L = \max\{\theta^n(a), \dots, \theta^{ln}(a)\}$. Hence

$$T^m \xi(a) = T^{n+m} \xi(a) = T^{m+2n} \xi(a) = \dots = T^{m+ln} \xi(a)$$

so therefore

$$\xi(\theta^m(a)) = \xi(\theta^{n+m}(a)) = \xi(\theta^{m+2n}(a)) = \dots = \xi(\theta^{m+ln}(a)).$$

If the common value of these expressions is j then, by the definition of ξ

$$\{\theta^m(a), \theta^{m+n}(a), \theta^{m+2n}(a), \dots, \theta^{m+ln}(a)\} \subset C_j.$$

■

Note that the theorem shows j depends on l . Since j has a finite range it is easy to deduce that j may be chosen to be the same for every l . The numbers m and n will then depend on j, l, a and, of course, θ .

3. APPLICATIONS

EXAMPLE 3.1. Let $\theta(n) \rightarrow n + 1$. This is Furstenberg's shift map. Using $a = 0$ we have $\theta^m(0) = m$ so van der Waerden follows. Using the notation in the theorem, there is a j so that for every l there is an m, n such that:

$$\{m, m + n, \dots, m + ln\} \subset C_j.$$

EXAMPLE 3.2. Let $\theta(n) \rightarrow 2n, a = 1$. Then $\theta^m(1) = 2^m$ so

$$\{2^m, 2^m \cdot 2^n, \dots, 2^m \cdot 2^{ln}\} \subset C_j,$$

that is to say a geometric progression of powers of 2.

EXAMPLE 3.3. Let $\theta(n) \rightarrow \alpha n + \beta, \alpha \geq 2, \beta \geq 1$. Then

$$\theta^m(n) = \alpha^m n + \beta \frac{\alpha^m - 1}{\alpha - 1}.$$

If we select $\alpha = 2, \beta = 1, a = 0$ then $\theta^m(0) = 2^m - 1$ and so:

$$\{2^m - 1, 2^{m+n} - 1, \dots, 2^{m+ln} - 1\} \subset C_j.$$

EXAMPLE 3.4. Let $A \subset \mathbb{N}$ be any infinite subset. Define θ_A by

$$\theta_A(n) = \min\{x \in A : x > n\}.$$

For example if $A = \mathbb{P}$ the rational primes, $\theta_{\mathbb{P}}(n)$ is the next prime not equal to n . The composite $\theta_A^m(0)$ is the m 'th term in A with increasing order. If we label the terms in order $A = \{a_i\}$, then by the theorem above,

$$\{a_m, a_{m+n}, a_{m+2n}, \dots, a_{m+ln}\} \subset C_j.$$

Applying this to $A = \mathbb{P}$ this means there is a C_j such that the m 'th, the $m + n$ 'th, up to the $m + ln$ 'th prime are in C_j .

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