Linear law for the logarithms of the Riemann periods at simple critical zeta zeros.

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Each simple zero $\frac{1}{2}+i\gamma_n$ of the Riemann zeta function on the critical line with $\gamma_n > 0$ is a centre for the flow $\dot{s} = \xi(s)$ of the Riemann xi function with an associated period $\tau_n = (-1)^{n+1}T_n$; $T_n > 0$. It is shown that, as $\gamma_n \to \infty$,

$$\log T_n \geq \frac{\pi}{4}\gamma_n + O(\log \gamma_n).$$

Numerical evaluation leads to the conjecture that this inequality can be replaced by an equality. Assuming the Riemann Hypothesis and a zeta zero separation conjecture we obtain the upper bound $\log T_n \ll \gamma_n^{2+\epsilon}$. The corresponding frequencies relate to natural eigenvalues for the Hilbert-Polya conjecture.

Key Words: Riemann zeta function, xi function, zeta zeros, periods, critical line, Hilbert-Polya conjecture

MSC2000: 11M06, 11M26, 11S40.

1. INTRODUCTION

If a holomorphic function of a single complex variable $f(s)$ has a simple zero at s_o which is a centre for the dynamical system $\dot{s} = f(s)$, then the period of an orbit encircling s_o is given by $T = 2\pi i/f'(s_o)$, [3, Theorem 2.3] and, in particular, is independent of the orbit. When this is applied to the simple zeros of Riemann's function $\xi(s)$, which lie on the critical line $s = \frac{1}{2}$, we see that the periods $\{\tau_j = 2\pi i/\xi'(\frac{1}{2}+i\gamma_j) : \xi(\frac{1}{2}+i\gamma_j) = 0\}$ could be of interest since each such (real) number applies to an infinite family of nested orbits.

Even though the positions of the zeros on the critical line have a considerable degree of random variation, the values of the periods are quite constrained, as is illustrated numerically in Figure 1 and partly proved in Theorem 3.1, the logarithm of the absolute value of the periods varies linearly with the position of the zero.

In Section 2 some preliminary results are listed. In Section 3 the numerical evaluation is developed. Of the 800 zeta zeros studied, being those with smallest positive t coordinate, the periods always increased with increasing zero value, with 4 exceptions. The numerical evidence demonstrates a relationship between the periods and the y-coordinate of the corresponding zero which is very close to being linear.

In Section 4 the relationship

$$\log T_n \geq \frac{\pi}{4}\gamma_n + O(\log \gamma_n)$$

where $T_n = (-1)^n \tau_n$, is proved. An upper bound is derived subject to the Riemann Hypothesis and a plausible conjecture on zero separations. In Section 5 a possible significance of the periods, in the context of the Hilbert-Polya conjecture, is sketched.

First some notation. Let $s = \sigma + it$, $\xi(s) = u(\sigma, t) + iv(\sigma, t)$. Label the simple zeros of $\xi(s)$, with positive t coordinate, going up the critical line as $\{\gamma_n : n \in \mathbb{N}\}$. A particular zero $\frac{1}{2} + i\gamma_n$ has the corresponding period

$$\tau_n = \frac{2\pi i}{\xi'(\gamma_n)},$$

so $T_n = 2\pi / |u_t(\frac{1}{2}, \gamma_n)|$.

Let $s = \frac{1}{2} + it$ and write

$$\begin{aligned}
\xi(\tfrac{1}{2} + it) &= e^{\log \Gamma(\frac{s}{2})} \pi^{-s/2} \frac{s(s-1)}{2} \zeta(s) \\
&= -e^{\Re \log \Gamma(\frac{s}{2})} \pi^{-1/4} (\frac{t^2 + 1/4}{2}) \times Z(t) \\
&= Y(t) \times Z(t)
\end{aligned}$$

where

$$\begin{aligned}
\log \Gamma(s+1) &= (s + \tfrac{1}{2}) \log s - s + c_1 + O(\tfrac{1}{s}) \\
Z(t) &= e^{i\vartheta(t)} \zeta(\tfrac{1}{2} + it) \\
\vartheta(t) &= \frac{t}{2} \log(\frac{t}{2\pi}) - \frac{t}{2} - \frac{\pi}{8} + O(\frac{1}{t})
\end{aligned}$$

and where the $c_1, c_2, \ldots$ are absolute constants.

2. PRELIMINARY LEMMAS

LEMMA 2.1. *As* $t \to \infty$

$$\Re \log \Gamma(\frac{\frac{1}{2}+it}{2}) = -\frac{1}{8}\log(9+4t^2) - \frac{\pi}{4}t + c_2 + O(\frac{1}{t}).$$

Proof. This follows directly from Stirling's approximation given above. ▌

LEMMA 2.2. *As* $t \to \infty$, $\vartheta'(t) = O(\log t)$.

Proof. This follows directly from the expression for $\vartheta(t)$ given above or see [8, Page 125]. ▌

LEMMA 2.3. *As* $t \to \infty$, $\zeta(\frac{1}{2}+it) = O(t^{1/6}\log^{3/2} t)$.

Proof. This is [13, Page 99]. ▌

LEMMA 2.4. *As* $t \to \infty$, $\zeta'(\frac{1}{2}+it) = O(t^{1/4}\log^{2} t)$.

Proof. This follows from the upper bound for $\zeta^{(k)}(\frac{1}{2}+it)$ derived using the approximate functional equation and Cauchy's integral formula for the n'th derivative given in [11, Page 57], namely

$$\begin{aligned} |\zeta^{(k)}(\tfrac{1}{2}+it)| \; &\le \; |\sum_{n\le\sqrt{t/2\pi}} \frac{\log^k n}{n^{1/2+it}}| + \log^k t \sum_{1\le j\le k} |\sum_{n\le\sqrt{t/2\pi}} \frac{\log^k n}{n^{1/2+it}}| \\ &+ \; O(t^{-1/4}\log^k t). \end{aligned}$$

▌

LEMMA 2.5. *As* $t \to \infty$, $Z'(t) = O(t^{1/4}\log^{2} t)$.

Proof. This follows directly from $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$ after differentiating and using Lemmas 2.2 and 2.4. ▌

3. NUMERICAL EVALUATION OF THE PERIODS

We used the Chebyshev method of P. Borwein [2]. Our version is described in [3, section 5] in which accuracies for the ζ-zeros better than 10^{-10} were demonstrated.

A program was written in MATLAB using the methods of Godfrey [10] for both the ζ-function and the complex Γ-function They are available in Godfrey's MATLAB suite as part of the zeta-function code. In particular, values of the gamma function needed for the definition:

$$\xi(\tfrac{1}{2}+it)=\tfrac{1}{2}s(s-1)\Gamma(\tfrac{s}{2})\pi^{-s/2}\zeta(s)$$

do not have to be incorporated into a Riemann-Siegel type formulation. Figure 1 gives an illustration of the linear relation of this paper for zeros up to $t=\gamma_{502}=814.1$.

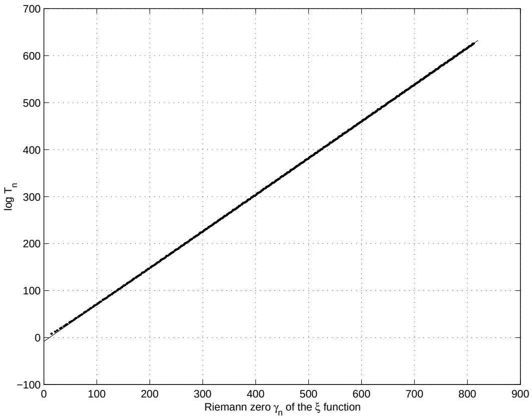

FIG. 1. Plot of logarithm of the periods against the Riemann zeros up to $t = 814.1$.

Figure 2 shows the deviations from the linear law $\log T_n=\alpha+\frac{\pi}{4}\gamma_n$ There is a rise at low γ-values. The value of α for the fit is -24.55.

Table 1 shows the zeros and periods; for $0<n\leq 20$ where the slope deviates slightly from $\pi/4$ as is seen in Figure 2. The *rms* deviation for the fit is 1.5.

Table 2 includes the zeros and periods for $\gamma_{300}=541.847$, $\gamma_{365}=632.225$, $\gamma_{447}=741.757$and $\gamma_{482}=787.468$, where, anomalously we find, $\log T_n < \log T_{n-1}$ and these are the only examples where the period decreases up to γ_{502}.

An alternative calculation was devised to extend the modest upper value of $t=800$ to $t=40000$. The same relationship was found to hold. The symbolic ζ-calculations of Maple^{TM} were combined with Godfrey's implementation of Lanczos' Γ-function method. Earlier values $t \leq 814.1$ were confirmed. The results are presented in blocks of 500 as shown in Figure 4.

n	γ_n	$\log T_n$	T_n
1	14.13473	8.4242	4.5562e+003
2	21.02204	12.7793	3.5480e+005
3	25.01086	15.4167	4.9589e+006
4	30.42488	19.3796	2.6089e+008
5	32.93506	21.1554	1.5406e+009
6	37.58618	24.2391	3.3645e+010
7	40.91872	26.9664	5.1445e+011
8	43.32707	28.5531	2.5145e+012
9	48.00515	32.2017	9.6613e+013
10	49.77383	33.6324	4.0399e+014
11	52.97032	35.4970	2.6071e+015
12	56.44625	38.1368	3.6525e+016
13	59.34704	40.8634	5.5814e+017
14	60.83178	41.8135	1.4434e+018
15	65.11254	44.7276	2.6604e+019
16	67.07981	46.4654	1.5124e+020
17	69.54640	48.1391	8.0639e+020
18	72.06716	49.7490	4.0339e+021
19	75.70469	53.0341	1.0775e+023
20	77.14484	54.3339	3.9529e+023

Table 1 Numerical values of the first 20 ξ-values and logarithms of the periods.

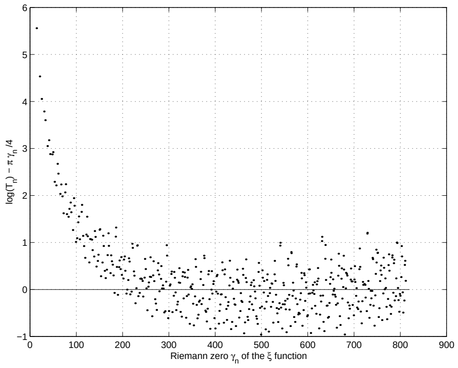

FIG. 2. The residuals using a slope of $\pi/4$.

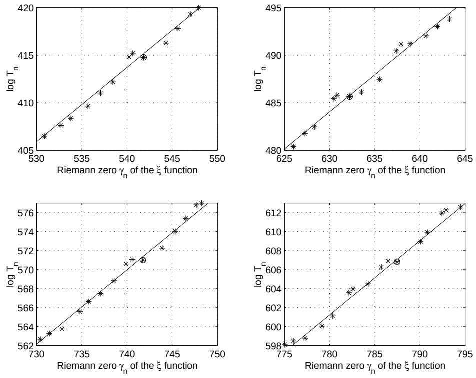

FIG. 3. The four anomalous deviations from monotonicity.

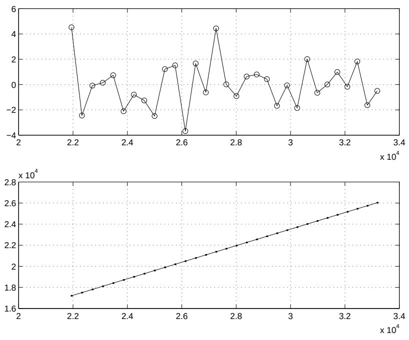

FIG. 4. The linear periods (below) and residuals using a slope of $\pi/4$.

n	γ_n	$\log T_n$	T_n
299	540.63139	415.2032	2.0914e+180
300	541.84744	414.7731	1.3605e+180
301	544.32389	416.2643	6.0437e+180
364	630.80578	485.7808	9.3738e+210
365	632.22514	485.6408	8.1491e+210
366	633.54686	486.1135	1.3075e+211
446	740.57381	571.0652	1.0244e+248
447	741.75734	571.0002	9.5995e+247
448	743.89501	572.2458	3.3357e+248
481	786.46115	606.9047	3.7614e+263
482	787.46846	606.8254	3.4747e+263
483	790.05909	608.9406	2.8810e+264

Table 2 Numerical values of the logarithms of the periods and zero ξ-values showing the 4, anomalous, decreasing periods as n increases.

4. LOWER AND UPPER BOUNDS FOR THE LINEAR RELATIONSHIP

Theorem 4.1. *For $n \in \mathbb{N}$*

$$\log T_n \geq \frac{\pi}{4}\gamma_n + O(\log \gamma_n).$$

Proof.

$$|u_t(\tfrac{1}{2}, t)| = |\xi'(\tfrac{1}{2} + it)|$$

$$i\xi'(\tfrac{1}{2} + it) = Y'(t)Z(t) + Y(t)Z'(t)$$

Since $Z(\gamma_j) = 0$,

$$|\xi'(\tfrac{1}{2} + i\gamma_j)| = |Y(\gamma_j)| \times |Z'(\gamma_j)|.$$

so,

$$\log|u_t(\tfrac{1}{2},\gamma_j)| = \log|Y(\gamma_j)| + \log|Z'(\gamma_j)|$$

By Lemma 2.1,

$$\begin{aligned}\log|Y(t)| &= \Re\log\Gamma(\frac{\frac{1}{2}+it}{2}) - \frac{1}{4}\log\pi + \log(\frac{1}{8}+\frac{t^2}{2})\\ &= -\frac{\pi}{4}t + \log\big(\frac{1+4t^2}{(9+4t^2)^{1/8}}\big) + c_3 + O(\frac{1}{t}).\end{aligned}$$

Therefore, by Lemma 2.5,

$$-\log|u_t(\tfrac{1}{2},\gamma_n)| \le \tfrac{\pi}{4}\gamma_n + O(\log\gamma_n).$$

and therefore

$$\log T_n \ge \frac{\pi}{4}\gamma_n + O(\log\gamma_n).$$

▮

Finding upper bounds for the periods appears to be a much more difficult problem. Assuming the Riemann Hypothesis (RH) and a plausible zero separation hypothesis we can make some progress on this issue:

Fuji [9] has shown that there exists a $\lambda > 1$ such that for n in a set of positive density

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \ge \lambda.$$

Since [13, Page 214] $\gamma_n \sim 2\pi n/\log n$, this implies that for all $\epsilon > 0$ and n sufficiently large and in the subset of positive density, $\gamma_{n+1}-\gamma_n \ge 1/\gamma_n^\epsilon$. Here we assume that for all $\epsilon > 0$ for all $n \in \mathbb{N}$

$$\gamma_{n+1}-\gamma_n \gg \frac{1}{\gamma_n^\epsilon}$$

where the implied constant depends on ϵ. Call this assumption the "zero separation conjecture" or ZSC. Of course if true it implies all zeros are simple. It supports the notion of Montgomery that the zeros tend to "repel", but does alow zeros to be very close.

Theorem 4.2. *Assume RH and ZSC. Then for all $n \in \mathbb{N}$:*

$$\log T_n \ll \gamma_n^{2+\epsilon}.$$

Proof. Using the shifted product representation for $\xi(s)$

$$\xi(z+\tfrac{1}{2}) = \xi(\tfrac{1}{2})\textstyle\prod_{n\in\mathbb{N}}(1+\frac{z^2}{\gamma_n^2})$$

leads to the value of the derivative at a zero

$$\begin{aligned}
u_t(\tfrac{1}{2},\gamma_n) &= \frac{2\xi(\frac{1}{2})}{\gamma_n}\prod_{j\in\mathbb{N},j\neq n}(1-\frac{\gamma_n^2}{\gamma_j^2}) \\
|u_t(\tfrac{1}{2},\gamma_n)| &= \frac{2\xi(\frac{1}{2})}{\gamma_n}\prod_{1\leq j<n}(\frac{\gamma_n^2}{\gamma_j^2}-1)\prod_{n<j}(1-\frac{\gamma_n^2}{\gamma_j^2}) \\
&= \frac{2\xi(\frac{1}{2})}{\gamma_n}X_n\times Y_n,\ \text{say.}
\end{aligned}$$

Because the number of zeros at level T is less than $T\log T/2\pi$ it follows that

$$X_n = \prod_{1\leq j<n}(\frac{\gamma_n+\gamma_j}{\gamma_j})(\frac{\gamma_n-\gamma_j}{\gamma_j}) \geq (\frac{\gamma_n-\gamma_{n-1}}{\gamma_n})^{\gamma_n\log\gamma_n}.$$

Therefore, by the assumption ZSC,

$$-\log X_n \ll (\epsilon+1)\gamma_n\log^2\gamma_n \quad (1).$$

Also, because $-\log(1-x)\leq x/(1-x)$ for $0\leq x<1$,

$$\begin{aligned}
Y_n &= \prod_{j>n}(1-\frac{\gamma_n^2}{\gamma_j^2}) \text{ so} \\
-\log Y_n &= \sum_{j>n}-\log(1-\frac{\gamma_n^2}{\gamma_j^2}) \leq \sum_{j>n}\frac{\gamma_n^2}{\gamma_j^2-\gamma_n^2} \quad (2).
\end{aligned}$$

But

$$\begin{aligned}
\sum_{j>n}\frac{1}{\gamma_j^2-\gamma_n^2} &= \sum_{\gamma_j^2<\frac{10}{9}\gamma_n^2}\frac{1}{\gamma_j^2-\gamma_n^2} + \sum_{\gamma_j^2\geq\frac{10}{9}\gamma_n^2}\frac{1}{\gamma_j^2-\gamma_n^2} \\
&= S_1+S_2 \text{ say.}
\end{aligned}$$

Since, in S_2, ${\gamma_j}^2-\gamma_n^2 \geq \gamma_j^2-\frac{9}{10}\gamma_j^2 = \frac{1}{10}\gamma_j^2$,

$$S_2 \leq \sum_{\gamma_j^2\geq\frac{10}{9}\gamma_n^2}\frac{10}{\gamma_j^2} \ll \frac{\log^2\gamma_n}{\gamma_n}.$$

The sum S_1 is finite with largest term being the first which is

$$\frac{1}{(\gamma_{n+1}+\gamma_n)(\gamma_{n+1}-\gamma_n)} \ll \frac{\gamma_n^\epsilon}{2\gamma_n}.$$

The number of terms is bounded by

$$\int_{\gamma_n}^{\sqrt{10/9}\gamma_n} t\log t\ dt = O(\gamma_n \log\gamma_n)$$

so

$$S_1 \ll \gamma_n^\epsilon \log\gamma_n$$

and therefore by (2)

$$-\log Y_n \ll \gamma_n^2(\gamma_n^\epsilon \log\gamma_n + \frac{\log^2\gamma_n}{\gamma_n}) \ll \gamma_n^{2+\epsilon}\ (3)$$

It follows from (1) and (3) that

$$\log T_n \ll -\log|u_t(\tfrac{1}{2},\gamma_n)| \ll \gamma_n^{2+\epsilon}.$$

▮

5. HILBERT-POLYA CONJECTURE

The well known approach to proving the Riemann Hypothesis, attributed to Polya and Hilbert [12] is that there is a naturally occurring Hermitian operator whose eigenvalues are the non-trivial zeros of $\xi(\frac{1}{2}+it)$, which are therefore real [7, Page 345]. There have been many attempts to find such an operator [1], but none so far has been completely successful. The normal interpretation of the heights of the zeros $\{\gamma_i : i \in \mathbb{N}\}$ is that they correspond to frequencies of some, to be determined, vibrating system.

Consideration of the phase portraits described in [5, 6] in the light of the constancy of the period for rotation about each simple zero on the critical line [3, Theorem 2.3], gives rise to the notion that an alternative set of potential eigenvalues might be related to these periods $\{T_i : i \in \mathbb{N}\}$ in the usual manner:

$$f_i = \frac{1}{T_i} = \frac{\lambda_i}{2\pi} = \frac{|u_t(\frac{1}{2},\gamma_i)|}{2\pi} \approx \frac{e^{-\frac{\pi\gamma_i}{4}}}{2\pi} \approx \frac{1}{2\pi}exp[-\frac{\pi\gamma_i}{4}]$$

so $\lambda_i \approx e^{-\frac{\pi\gamma_i}{4}}$.

This approach has the following features:

(a) The eigenvalues have a natural relationship to $\zeta(s)$ and $\xi(s)$ - true also of course of the zeta zeros.

(b) The eigenvalues are the natural frequencies occurring in the flow $\dot{s} = \xi(s)$.

(c) The lowest zero corresponds to the largest eigenvalue and therefore to the mode of highest energy and largest frequency. This is quite natural, since the values of $\xi(s)$ with s real and with values which are large and positive might be regarded as exerting a strong attractive pull on the flow, and would be expected to generate the greatest energy in the closest pair of zeros, which corresponding frequencies become the "fundamental" modes. This "attractor" concept is strongly reinforced by the structure of the separatrices which all tend to the real axis, [5, Theorem 3.2]. This is much more satisfactory than the normal approach in which the energy increases as the distance away from the real axis increases.

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