ASYMPTOTIC ORDER OF THE SQUARE FREE PART OF N!

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Abstract

A recursive formula is derived to calculate the square free part of $n!$ which appears graphically to have an approximately linear variation for its logarithm. Using this the asymptotic order of the logarithm of the square free part of $n!$ is then shown to be $(\log 2)n$ with error $O(\sqrt{n})$.

1. Introduction

If the standard prime factorization of $n!$ is considered over a range of values of n then a number of patterns are apparent:

$$\begin{aligned} 10! &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \\ 20! &= 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \\ 30! &= 2^{26} \cdot 3^{14} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\ 40! &= 2^{38} \cdot 3^{18} \cdot 5^9 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37. \end{aligned}$$

All the primes up to n appear. If p and q are primes appearing in the factorization with $p < q$ and α, β are the highest powers of p and q dividing $n!$ respectively, then $\alpha \geq \beta$, i.e. the smaller the prime, the larger the power. Even though sometimes a given power does not appear (the power 3 is missing from 20! even though the powers 2 and 4 appear), the power 1 always appears.

Primes which appear to power 1 divide the square free part of $n!$. This is the number a, with no square factors, which appears in the factorization

$$n! = ab^2.$$

It is easy to see that a is exactly the product of each of the primes which appear to an odd power in the standard factorization.

Two natural questions arise: what is the size of the square free part a of $n!$ and what proportion of a is the product of the primes which occur to power 1 ? In this note it will be shown that, asymptotically, the proportion of primes to power 1 is about 72%. If the primes which appear to power 3 are included also then the figure is about 84%.

2. Integer Square Roots

For each whole number n let the integer lower square root be defined by

$$r_-(n) = \prod_{p^\alpha || n} p^{\lfloor \frac{\alpha}{2} \rfloor}$$

and the integer upper square root by

$$r_+(n) = \prod_{p^\alpha || n} p^{\lceil \frac{\alpha}{2} \rceil}$$

If $n = ab^2$ and $cn = d^2$ with a and c square free, then $ac = m^2$ where m is square free, being the product of the primes in n which appear to odd powers. For each $n \in \mathbb{N}$ the integers a, b, c, d and m are uniquely determined with m square free, $b = r_-(n)$, $d = r_+(n)$ and $n = r_+(n) \cdot r_-(n)$. Finally

$$a = c = m = \prod_{p|n} p^{\lceil \frac{\alpha}{2} \rceil - \lfloor \frac{\alpha}{2} \rfloor}.$$

This pair of functions $r_\pm$ is quite useful. They are multiplicative, can be generalized to integer k'th roots and are related to the integer conductor or square free core. For examples and applications see the articles [3, 4].

3. Computing the square free part of n!

To obtain some idea of the behavior of the square free part of $n!$, for large n, is pays to do some computations. However for numbers of quite small size, say $n = 200$ or $n = 400$, $n!$ is a very large number. In the latter case the number has over 800 digits, so finding the square free part should not be attempted directly. The following strategy was adopted:

For each $n \geq 1$, let θ_n be the square free part of $n + 1$, i.e.,

$$\theta_n = r_+(n + 1)/r_-(n + 1).$$

Because $a_{n+1}b_{n+1}^2 = (n + 1)n! = (n + 1)a_n b_n^2$ and $n + 1 = \theta_n c^2$ for some integer c, we have $\theta_n a_n b_n^2 = a_{n+1}b_{n+1}^2$.

If a prime $p \mid (\theta_n, a_n)$, then p occurs as a factor in both θ_n and a_n, so must occur to an odd power in both $n!$ and $n + 1$, and therefore to an even power in $(n + 1)!$. Hence it does not occur in a_{n+1}. If a prime occurs in just one of θ_n and a_n, then it must occur in a_{n+1}. This leads directly to the formula:

$$a_{n+1} = \frac{a_n \theta_n}{(a_n, \theta_n)^2}. \tag{1}$$

Note that this formula can be used to evaluate the sequence (a_n) recursively, so the values of $\log a_n$ can be plotted, revealing a nice approximately linear dependence on n. See Figure 1.

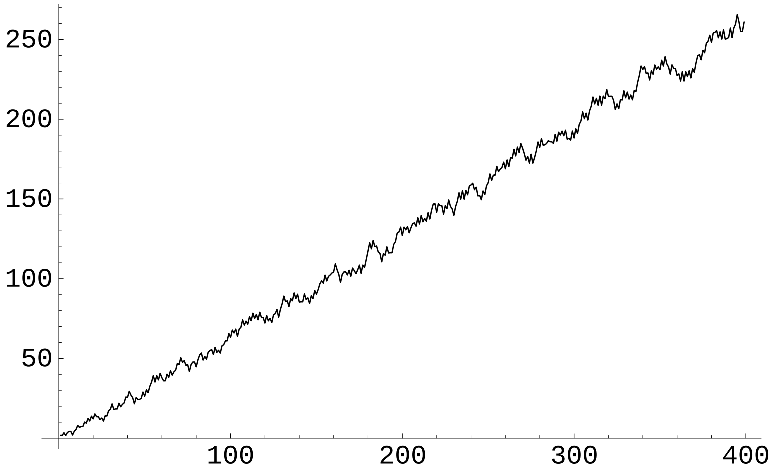

Figure 1: The sequence $\log a_n$ as a function of n.

4. Asymptotic orders

The result of these computations of the square free part of $n!$ lead to two natural tasks: determining the slope of a line approximating the graph of $\log a_n$, and finding an upper bound for the error in this approximation. The completion of both tasks is summarized in the next theorem.

Theorem 1: For each $n \in \mathbb{N}$ let $n! = a_n b_n^2$ where $a_1 = b_1 = 1$ and where for all $n \geq 1$, a_n is square free.

Then $$\log a_n = n \log 2 + O(\sqrt{n}),$$

and $$\log b_n = \frac{1}{2} n \log n - \frac{1 + \log 2}{2} n + O(\sqrt{n}).$$

Proof:

Consider the central binomial coefficient $\binom{2n}{n} = t_n s_n^2$ where t_n is square free. Then

$$b_{2n}^2 a_{2n} = (2n)! = (n!)^2 s_n^2 t_n$$

so $t_n = a_{2n}$ for all $n \in N$. By the main result in [7], there is a real positive constant c such that for all $\epsilon > 0$ and all n sufficiently large

$$c - \epsilon\sqrt{n} < 2 \log s_n < c + \epsilon\sqrt{n}.$$

Therefore $2\log s_n = O(\sqrt{n})$.

The Stirling's series approximation for $n!$ as $n \to \infty$ can be referenced on the web at [8]. It is

$$n! = \sqrt{2\pi n}\big(\frac{n}{e}\big)^n\big(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{5140n^3}+O(\frac{1}{n^4})\big).$$

It leads to the formula:

$$\log n! = n\log n - n + O(\log n)$$

so
$$\begin{aligned}\log\binom{2n}{n} &= 2n\log 2n - 2n - 2n\log n + 2n + O(\log n)\\ &= 2n\log 2 + O(\log n)\\ &= \log a_{2n} + 2\log s_n\\ &= \log a_{2n} + O(\sqrt{n}).\end{aligned}$$

Hence
$$\log a_{2n} = 2n\log 2 + O(\sqrt{n}).$$

By equation (1)
$$\begin{aligned}\log a_{2n+1} &= \log a_{2n} + \log\theta_{2n} - 2\log(a_{2n},\theta_{2n})\\ &= \log a_{2n} + O(\log n) \text{ since } \theta = O(n)\\ &= (2n+1)\log 2 + O(\sqrt{n})\end{aligned}$$

and therefore
$$\log a_n = n\log 2 + O(\sqrt{n}).$$

But, by Stirlings approximation again and this estimate for $(\log a_n)$:

$$\begin{aligned}2\log b_n &= n\log n - n - n\log 2 + O(\sqrt{n})\\ &= n\log n - (1+\log 2)n + +O(\sqrt{n})\end{aligned}$$

and therefore $\log b_n = \frac{1}{2}n\log n - \frac{1+\log 2}{2}n + O(\sqrt{n})$. This completes the proof of the theorem.

It follows also that the square free part of $\binom{2n}{n}$, namely t_n, satisfies $\log t_n = 2n\log 2 + O(\sqrt{n})$, giving the asymptotic order. This relates to the solved conjecture of Erdös [5] that the binomial coefficient $\binom{2n}{n}$ is not square free for $n > 4$. It relates also to the parity of the exponents of the prime factors of $n!$, [2].

5. Primes dividing n!

A nice formula of Legendré enables the maximum power $\alpha = alpha_p$ of a prime p dividing $n!$ to be calculated, namely

$$\alpha_p = \sum_{j=1}^{\infty}\lfloor\frac{n}{p^j}\rfloor.$$

This will be used below to describe the primes which appear in the factorization of $n!$ to the first and third powers.

Lemma 1: Let $n \geq 1$ and let p be a prime integer.

(a) Let $\lfloor \frac{n}{p} \rfloor \leq 3$ and$\lfloor \frac{n}{p^2} \rfloor > 0$. Then $\lfloor \frac{n}{p^2} \rfloor = 1$ and $p \in \{2, 3\}$.

(b) If $n \geq 2$ then $p^1 \| n!$ if and only if $\frac{n}{2} < p \leq n$.

(c) If $n \geq 12$ then $p^3 \| n!$ if and only if $\frac{n}{4} < p \leq \frac{n}{3}$.

Proof

(a) If $\lfloor \frac{n}{p^2} \rfloor = u$ then $u \leq \frac{n}{p^2} < u + 1$ so $up \leq n/p$. Then $up \leq \lfloor \frac{n}{p} \rfloor \leq 3$ and it follows directly that $u = 1$ and $p \in \{2, 3\}$.

(b) By the formula of Legendré quoted above, if $p^1 \| n!$ then $\lfloor \frac{n}{p} \rfloor = 1$ and $\lfloor \frac{n}{p^2} \rfloor = 0$. Hence $1 \leq \frac{n}{p} < 2$ which means $\frac{n}{2} < p \leq n$ and $n < p^2$. Therefore $\sqrt{n} < n$ so $1 < n$. Conversely if $\frac{n}{2} < p \leq n$ then $n < 2p \leq p^2$ so $p^1 \| n!$.

(c) There are three cases to consider: when $\lfloor \frac{n}{p} \rfloor = 3$ or when $\lfloor \frac{n}{p} \rfloor = 2$ and $\lfloor \frac{n}{p^2} \rfloor = 1$ or when $\lfloor \frac{n}{p} \rfloor = \lfloor \frac{n}{p^2} \rfloor = \lfloor \frac{n}{p^3} \rfloor = 1$ with all other $\lfloor \frac{n}{p^j} \rfloor = 0$. In the second and third case, $\lfloor \frac{n}{p^2} \rfloor > 0$ so by (a), these cases do not occur if $n \geq 12$. Hence we need only consider the first case. But then $\alpha_p = \lfloor \frac{n}{p} \rfloor = 3$ holds if and only if $3 \leq \frac{n}{p} < 4$ which is equivalent to $\frac{n}{4} < p \leq \frac{n}{3}$. This completes the proof of the lemma.

To evaluate the logarithm of the product of primes in a range we will use Chebychev's function $\theta(x)$ defined for $x > 0$ by

$$\theta(x) = \sum_{2 \leq p \leq x} \log p,$$

where the sum is over all primes less than or equal to x. The statement $\lim_{x \to \infty} \frac{\theta(x)}{x} = 1$ as $x \to \infty$ is equivalent to the Prime Number Theorem [1].

If $x \geq 563$ then $\theta(x)$ is close to x in that [6]

$$x(1 - \frac{1}{2 \log x}) < \theta(x) < x(1 + \frac{1}{2 \log x}).$$

By Lemma 1, part (a), the logarithm of the product of primes which appear in $n!$ to the first power is

$$\begin{aligned} \log \prod_{\frac{n}{2} < p \leq n} p &= \sum_{\frac{n}{2} < p \leq n} \log p \\ &= \theta(n) - \theta(\frac{n}{2}) \\ &= \frac{n}{2} + O(\frac{n}{\log n}), \end{aligned}$$

so asymptotic order of the product is $\frac{n}{2}$ as $n \to \infty$.

By Lemma 1, part (c), the logarithm of the product of primes which appear in $n!$ to the third power is given by

$$\sum_{\frac{n}{4}<p\leq\frac{n}{3}} \log p = \frac{n}{12} + O(\frac{n}{\log n}).$$

By Theorem 1, the asymptotic order of the square free part of $n!$ is $(\log 2)n$. Hence primes to power one contribute $\frac{1/2}{\log 2}$ or about 72%, and those to power one or three to $\frac{1/2}{\log 2}$, or about 84% of the square free part.

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