Extension of the Riemann ξ-function's logarithmic derivative positivity region to near the critical strip

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Kevin A. Broughan

Department of Mathematics, University of Waikato,
Hamilton, New Zealand
E-mail: kab@waikato.ac.nz

The inequality

$$\Re\frac{\xi'(\sigma+\mathrm{i}t)}{\xi(\sigma+\mathrm{i}t)} > \frac{\xi'(\sigma)}{\xi(\sigma)}$$

for $t \neq 0$ is extended to the region $\sigma \geq 1 + 1/(\log t - 5)$ for all $t \neq 0$ and for $\sigma \geq 1$ for t sufficiently large or small.

Key Words: Riemann zeta function, xi function, zeta zeros.

MSC2000: 11M26, 11R42.

1. INTRODUCTION

In the paper [3] Lagarias shows that, assuming the Riemann hypothesis,

$$\Re\frac{\xi'(\sigma+\mathrm{i}t)}{\xi'(\sigma+\mathrm{i}t)} > \frac{\xi'(\sigma)}{\xi(\sigma)}$$

for all $\sigma > 1/2$ and for all $t \neq 0$. He also shows that this inequality holds unconditionally in case $\sigma \geq 10$ and remarks that it seems likely the inequality could be established unconditionally for $\sigma > 1 + \epsilon$ for any given fixed positive ϵ "by a finite computation". Here we derive the inequality unconditionally up to $\sigma = 1$ for t sufficiently small or large, and for mid-range t to $\sigma \geq 1 + 1/(\log|t| - 5)$. This is Theorem 3.1, proved following 5 lemmas. "Sufficiently small" means up-to a value of t which satisfies $|t| \leq \sqrt{2-\sqrt{2}}\gamma$, where γ is the y-coordinate of the first off critical line non-trivial zero of $\zeta(s)$. "Sufficiently large" means greater than $e^{(e^{16c_1^3})}$

where c_1 is the absolute constant appearing in the an inequality for the logarithmic derivative of $\zeta(s)$.

2. PRELIMINARY LEMMAS

LEMMA 2.1. *There exists an absolute constant c_1 such that for all $\sigma \geq 1$ and all $t \geq t_1 > 0$*

$$|\frac{\zeta'(s)}{\zeta(s)}| \leq c_1 (\log t)^{2/3} (\log \log t)^{1/3}.$$

Proof. This follows from Richert [5] or Cheng [1]. See also [6, Section 6.19]. ∎

LEMMA 2.2. *For $\sigma > 1$ let*

$$f(\sigma) := \frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{\sigma - 1} - \gamma_o.$$

where γ_o is Euler's constant. Then there exists a positive absolute constant c_2 such that

$$-c_2(\sigma - 1) < f(\sigma) < 0,$$

and c_2 can be taken to be $\gamma_o^2 - 2\gamma_1$ where

$$\gamma_1 = - \lim_{N \to \infty} \big(\sum_{m=2}^{N} \frac{\log m}{m} - \frac{\log^2 N}{2} \big) = -0.07235..$$

so $c_2 = 0.47789...$

Proof. Write

$$f(\sigma) = \frac{1}{\sigma - 1} - \gamma_o - \sum_{p, m \geq 1} \frac{\log p}{p^{m\sigma}}$$

so

$$\begin{aligned} f'(\sigma) &= \sum_{p,m\geq 1} \frac{m\log^2 p}{p^{m\sigma}} - \frac{1}{(\sigma-1)^2} \\ &= \sum_p \log^2 p \sum_{m\geq 1} \frac{m}{(p^\sigma)^m} - \frac{1}{(\sigma-1)^2} \\ &= \sum_p \frac{\log^2 p.p^\sigma}{(p^\sigma-1)^2} - \frac{1}{(\sigma-1)^2}. \end{aligned}$$

Hence

$$\begin{aligned} f''(\sigma) &= \frac{2}{(\sigma-1)^3} + \sum_p \frac{\log^3 p.p^\sigma}{(p^\sigma-1)^2} - 2\sum_p \frac{\log^3 p.p^\sigma}{(p^\sigma-1)^3} \\ &= \frac{2}{(\sigma-1)^3} + \frac{\log^3 2.2^\sigma(2^\sigma-3)}{(2^\sigma-1)^3} + \sum_{p\geq 3} \frac{\log^3 p.p^\sigma(p^\sigma-3)}{(p^\sigma-1)^3}. \end{aligned}$$

If $\sigma \geq \log_2 3$ each term is non-negative so $f''(\sigma) > 0$. If $1 < \sigma < \log_2 3$ the sum of the first two terms is positive, so in all cases $f''(\sigma) > 0$. Hence $f(\sigma)$ is concave upwards on $(1, \infty)$.

Now the Laurent expansion of $\zeta(s)$ in the neighborhood of $s = 1$ [2, Theorem 1.4] is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + ...$$

where, for $k \geq 0$

$$\gamma_k = \frac{(-1)^k}{k!} \lim_{n\to\infty} \Big(\sum_{m=1}^{n} \frac{\log^k m}{m} - \frac{\log^{k+1} n}{k+1} \Big).$$

so γ_0 is Euler's constant and $\gamma_1 < 0$. From this it follows that, in a neighborhood of $s = 1$,

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)} + \frac{1}{\sigma-1} - \gamma_o = (2\gamma_1 - \gamma_0^2)(s-1) + O((s-1)^2)$$

so $f'(1) = 2\gamma_1 - \gamma_0^2$. Therefore, by the concavity of $f(\sigma)$, $f(\sigma) > (2\gamma_1 - \gamma_0^2)(\sigma - 1)$ for $\sigma > 1$.

Now, by continuous extension, $f(1) = 0$ and $f'(1) < 0$. If there was a value $\sigma > 1$ with $f(\sigma) \geq 0$ then, by Rolle's theorem, there would be a value with $f'(\sigma)=0$ and so, since $\lim_{\sigma\to\infty} f(\sigma) = -\gamma_0$, a point with $f''(\sigma)=0$. But by what we have proved this is impossible. Hence $f(\sigma)<0$ for all $\sigma > 1$. ▮

LEMMA 2.3. *If $1 \le \sigma < 10$ and $t \ge t_2 > 0$:*

$$\Re\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)} \ge \log\frac{t}{2} - 2 - \frac{2}{5t^2}.$$

Proof. This follows directly using the asymptotic expression

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + R, |R| \le \frac{1}{10|z|^2}, |z| \ge 2, \Re z > 0.$$

and the bound

$$|\frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)}| \le 2$$

which holds for $1 \le \sigma \le 10$. ▮

LEMMA 2.4. *Let $\sigma \ge 1$, $0 < \beta < 1/2, \gamma > 0$ be real numbers and define*

$$h(t) := \frac{(\gamma-t)^2 + (\sigma-1/2)^2 - (1/2-\beta)^2}{((\sigma-\beta)^2+(\gamma-t)^2)((\sigma+\beta-1)^2+(\gamma-t)^2)},$$

and $f(t) := h(t) + h(-t)$. Let $c_4 = \sqrt{2-\sqrt{2}}$. Then for all t with $|t| < c_4\gamma$, $f(t) > f(0)$.

Proof. Define $u := \sigma - 1/2$ and $v := \sigma + \beta - 1$. Then $u > v > 0$ and we can write

$$\begin{aligned} h(t) &= \frac{(\gamma-t)^2+uv}{((\gamma-t)^2+u^2)((\gamma-t)^2+v^2)} \\ &= \frac{1}{u+v}\Big(\frac{u}{(\gamma-t)^2+u^2} + \frac{v}{(\gamma-t)^2+v^2}\Big). \end{aligned}$$

Then

$$\begin{aligned} f(t) &= h(t)+h(-t) \\ &= \frac{1}{u(u+v)}\Big(\frac{1}{(\gamma-t)^2/u^2+1} + \frac{1}{(\gamma+t)^2/u^2+1}\Big) \\ &\quad + \frac{1}{v(u+v)}\Big(\frac{1}{(\gamma-t)^2/v^2+1} + \frac{1}{(\gamma+t)^2/v^2+1}\Big). \end{aligned}$$

Let

$$g_\gamma(t) := \frac{1}{(\gamma - t)^2 + 1} + \frac{1}{(\gamma + t)^2 + 1}.$$

Then the derivative

$$g'_\gamma(t) = \frac{4t(-t^4 - 2(1+\gamma^2)t^2 + (3\gamma^4 + 2\gamma^2 - 1))}{((\gamma - t)^2 + 1)^2((\gamma + t)^2 + 1)^2}$$

and $g'_\gamma(0) = 0$, g'_γ is an odd function of t, and the numerator is positive if

$$0 < t < (\gamma^2 + 1)^{1/4}(2\gamma - (\gamma^2 + 1)^{1/2})^{1/2},$$

or for the slightly smaller but more convenient range $0 < t < \sqrt{2 - \sqrt{2}}\gamma = c_4\gamma$, so $g'_\gamma(t) > 0$ in this range. Hence

$$f'(t) = \frac{1}{u^2(u+v)} g'_{\frac{\gamma}{u}}(\frac{t}{u}) + \frac{1}{v^2(u+v)} g'_{\frac{\gamma}{v}}(\frac{t}{v})$$

is positive for t with $0 < t/u < c_4\gamma/u$ and $0 < t/v < c_4\gamma/v$, that is the same range as before. Therefore $f(0) < f(t)$. But $f(t)$ is even, so the same inequality holds for t negative also. ∎

Lemma 2.5. *Let c_0 be a positive real number representing the y coordinate of the first zeta zero which is off the critical line (assuming such a zero exists). If $0 < t < c_4 c_0$ and $1 \leq \sigma < 10$, then*

$$\Re\frac{\xi'(\sigma + \mathrm{i}t)}{\xi(\sigma + \mathrm{i}t)} > \frac{\xi'(\sigma)}{\xi(\sigma)}$$

Proof. With the same notation as in Levinson and Montgomery [4], we can write

$$\begin{aligned}
\Re\frac{\xi'(\sigma+\mathrm{i}t)}{\xi(\sigma+\mathrm{i}t)}-\frac{\xi'(\sigma)}{\xi(\sigma)} &= (\sigma-1/2)(I(\sigma,t)-I(\sigma,0)), where \\
I(\sigma,t) &= T_o+T_1, where \\
T_o &= \sum_{\beta<1/2}\Big[\frac{(\gamma-t)^2+(\sigma-1/2)^2-(1/2-\beta)^2}{((\sigma-\beta)^2+(\gamma-t)^2)((\sigma+\beta-1)^2+(\gamma-t)^2)} \\
&- \frac{\gamma^2+(\sigma-1/2)^2-(1/2-\beta)^2}{((\sigma-\beta)^2+\gamma^2)((\sigma+\beta-1)^2+\gamma^2)}\Big] \\
T_1 &= \sum_{\beta=1/2}\Big[\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2} \\
&- \frac{1}{(\sigma-1/2)^2+\gamma^2}\Big].
\end{aligned}$$

The proof of Lagarias [3], assuming the Riemann Hypothesis, shows that $T_1 > 0$ whether or not the Riemann hypothesis is assumed to be true. Lemma 2.4 shows that, since each $\gamma \geq c_0$, each term in the sum is positive for $t < c_4 c_0$ so the Lemma follows directly. ▮

3. PROOF OF THEOREM 3.1

THEOREM 3.1. *Let* $1 \leq \sigma < 10$ *and* $t \neq 0$. *Then there exist absolute constants* c_1, c_2 *so that (unconditionally) for* $\{\sigma + \mathrm{i}t : |t| \leq c_1, 1 \leq \sigma < 10\}$ *or* $\{\sigma + \mathrm{i}t : |t| \geq c_2, 1 \leq \sigma < 10\}$ *or* $\{\sigma + \mathrm{i}t : \log |t| \geq 5 + 1/(\sigma - 1), 1 < \sigma < 10\}$, *we have*

$$\Re\frac{\xi'(\sigma+\mathrm{i}t)}{\xi(\sigma+\mathrm{i}t)} > \Re\frac{\xi'(\sigma)}{\xi(\sigma)}$$

.

Proof. Let $s = \sigma + \mathrm{i}t$ and $1 < \sigma$. Then since

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)/2,$$

we can write

$$\begin{aligned}
\Delta_0 &:= \Re\frac{\xi'(s)}{\xi(s)} - \frac{\xi'(\sigma)}{\xi(\sigma)} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4, where \\
\Delta_1 &:= -\frac{t^2}{\sigma(\sigma^2+t^2)}, \\
\Delta_2 &:= -\frac{t^2}{(\sigma-1)((\sigma-1))^2+t^2)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)}, \\
\Delta_3 &:= \Re\frac{\zeta'(s)}{\zeta(s)}, \\
\Delta_4 &:= \frac{1}{2}\big(\Re\frac{\Gamma'(s/2)}{\Gamma(s/2)} - \frac{\Gamma'(\sigma/2)}{\Gamma(\sigma/2)}\big).
\end{aligned}$$

Firstly $\Delta_1 > -1/\sigma$. From Lemma 2.2 it follows that

$$\Delta_2 \geq -\gamma_0 + \frac{\sigma-1}{t}.$$

By Lemma 2.1 we can write

$$\Delta_3 = \Re\frac{\zeta'(s)}{\zeta(s)} \geq -c_1(\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}.$$

By Lemma 2.3 we can write

$$\Delta_4 = \log t - c_5\theta$$

for some small positive constant c_5 (we can take $c_5 = 4$) and real θ with $|\theta| < 1$.

Hence $\Delta_0 > 0$ if

$$\log t - c_1(\log t)^{2/3}(\log\log t)^{1/3} > 4,$$

This is true if and only if

$$1 - c_1\big(\frac{\log\log t}{\log t}\big)^{\frac{1}{3}} > \frac{4}{\log t}.$$

If we assume $t \geq t_3 := e^8$ then $1 - 4/\log t \geq 1/2$, so with this restriction we require

$$\frac{\log\log t}{\log t} < \frac{1}{8c_1^3}$$

This inequality holds if $t \geq t_4 := e^{(e^{16c_1^3})}$.

So provided $\gamma_0 c_4 \geq t_4$ the two regions $(0, \gamma_0 c_4], [t_4, \infty)$ overlap and the Lagarias inequality holds for all $\sigma > 1$. If however $t_4 > \gamma_0 c_4$ we argue differently. First let

$$\begin{aligned}
\Delta_2' &:= -\frac{t^2}{(\sigma-1)((\sigma-1))^2+t^2)}, \\
\Delta_3' &:= \Re\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta'(\sigma)}{\zeta(\sigma)}, so \\
\Delta_0 &= \Delta_1 + \Delta_2' + \Delta_3' + \Delta_4.
\end{aligned}$$

Since $|\frac{\zeta'(s)}{\zeta(s)}| \leq -\frac{\zeta'(\sigma)}{\zeta(\sigma)}$, $\Delta_3' \geq 0$ for all $t \geq 0$. Therefore

$$\Delta_0 > -\frac{1}{\sigma} - \frac{1}{\sigma-1} + \log t - 4,$$

so $\Delta_0 > 0$ if $\log t > 4 + 1/\sigma + 1/(\sigma-1)$ and this is true if $\log t > 5 + 1/(\sigma-1)$. The best uniform value of σ which may be obtained using this method is given approximately by

$$\sigma_0 = 1 + \frac{1}{\log c_4 c_0 - 5}.$$

∎

If we assume $c_4 c_0 = 10^8$ this leads to $\sigma_0 = 14/13$. Strengthening of the above approach to the Lagarias problem requires the derivation of a good explicit value for the constant c_1 (compare [1]) and knowledge of the best current value value for c_0 (currently 3.2×10^9) [7].

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