

# Adic Topologies for the Rational Integers

DRAFT:20 November 2001

Kevin A. Broughan

*University of Waikato, Hamilton, New Zealand*

E-mail: kab@waikato.ac.nz

A topology on  $\mathbb{Z}$ , which gives a nice proof that the set of prime integers is infinite, is characterised and examined. It is found to be homeomorphic to  $\mathbb{Q}$ , with a compact completion homeomorphic to the Cantor set. It has a natural place in a family of topologies on  $\mathbb{Z}$ , which includes the  $m$ -adics, and one in which the set of rational primes  $\mathbb{P}$  is dense. Examples from number theory are given, including the primes and squares, Fermat numbers, Fibonacci numbers and  $k$ -free numbers.

*Key Words:*  $p$ -adic,  $m$ -adic,  $m$ -topology metrizable, quasi-valuation, topological ring, completion, inverse limit, diophantine equation, Fermat numbers, Fibonacci numbers.

MSC2000 11B05, 11B25, 11B50, 13J10, 13B35.

## 1. INTRODUCTION

There is a nice topology on  $\mathbb{Z}$ , highlighted in reference [2], which enables a very elegant proof to be given that the number of rational primes is infinite. In this paper we develop properties of this topology, define a class of metrics which generate it, establish a natural family of topologies (the adic topologies) of which this is the finest and which includes the  $p$ -adic topologies, and give some examples from number theory.

The motivation behind this work is to provide some tools which will assist with the description and comparison of sets of integers, which are of number theoretic interest.

## 2. TOPOLOGIES FOR $\mathbb{Z}$

Definition of  $(\mathbb{Z}, \tau)$ : for each  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}$  with  $b \geq 1$ , let

$$N_{a,b} = \{a + nb : n \in \mathbb{Z}\}$$

Then for each  $a$  and  $b_1 \geq 1, b_2 \geq 1$

$$N_{a,b_1 b_2} \subset N_{a,b_1} \cap N_{a,b_2}$$

so the family  $(N_{a,b})$  is a base for the neighbourhoods of each point  $a$  and generates a topology,  $\tau$  on  $\mathbb{Z}$ , called here the full topology.

Now generalize this idea. For each  $a \in \mathbb{Z}$  let  $G_a$  be a multiplicative sub-semigroup of  $\mathbb{N}$  with 1 and let  $\mathcal{G} = (G_a : a \in \mathbb{Z})$ . Then let  $\tau_{\mathcal{G}}$  be the topology on  $\mathbb{Z}$  generated by  $\mathcal{B} = (N_{a,b} : a \in \mathbb{Z}, b \in G_a)$ , which is a sub-base.

Then  $\mathcal{G} \subset \mathcal{G}'$  implies  $\tau_{\mathcal{G}} \subset \tau_{\mathcal{G}'}$  so  $\tau_{\mathcal{G}} \subset \tau$  for all families  $\mathcal{G}$ . Therefore, in this class of topologies on  $\mathbb{Z}$ ,  $\tau$ , with  $G_a = \mathbb{N}$  for all  $a$ , is the finest, hence the name “full”.

We call topologies in this family “adic” topologies.

If  $G_a = \{1\}$  for each  $a$ , we obtain the indiscrete topology. In what follows, by the term semigroup we mean a sub multiplicative semigroup of  $\mathbb{N}$  with 1 which, unless otherwise stated, is non-trivial in that it contains an element  $b > 1$ .

Let  $p$  be a rational prime and, for each  $a \in \mathbb{Z}$ , let  $G_a$  be the semigroup generated by  $p$ , i.e.  $G_a = \{p^n : n = 0, 1, 2, \dots\}$ . Then  $\tau_{\mathcal{G}}$  is the classical p-adic topology.

Let  $m \in \mathbb{N}$  be fixed with  $m \geq 1$ . Define, for each  $a \in \mathbb{Z}$ ,

$$G_a = \{b \geq 1 : (a, b) = m^l \text{ for some integer } l \geq 0\}$$

Then  $\mathcal{G} = (G_a)$  generates  $\tau_m$ , the so called m-topology. Here  $G_a$  depends on  $a$ .

All topologies in this class make the multiplication  $\cdot$  continuous.

DEFINITION 2.1. If  $G_a = G$  is independent of  $a \in \mathbb{Z}$ , we say  $\mathcal{G}$  is flat.

DEFINITION 2.2. We say the semigroup  $G$  is divisor dense if for all  $n \in \mathbb{N}$  there is a  $b \in G$  such that  $n \mid b$ . If each  $G_a$  is divisor dense we say  $\mathcal{G}$  is divisor dense.

DEFINITION 2.3. If the semigroup  $G$  is such that  $a \in G$  and  $b \mid a$  implies  $b \in G$  we say  $G$  is divisor complete. This is equivalent to  $G$  being

generated by its prime elements. If each  $G_a$  is divisor complete we say  $\mathcal{G}$  is divisor complete.

Flat topologies make addition  $+$  continuous. If  $\mathcal{G}$  is divisor dense, then  $\mathcal{B}$  is a base for  $\tau_{\mathcal{G}}$ .

The semigroup collection  $\mathcal{G}$  which generates the topology  $\tau_m$  is neither divisor complete nor flat. However the corresponding  $\mathcal{B}$  is still a base. To see this let  $m > 1$  and let  $x \in N_{a,b} \cap N_{a',b'} = N_{x,b} \cap N_{x,b'}$ . Let  $(x, b) = m^r$  and  $(x, b') = m^{r'}$ . There are positive integers  $r, r'$  such that  $\beta m^r = b$  and  $\beta' m^{r'} = b'$ . Therefore  $(\frac{x}{m^r}, \beta) = 1$ ,  $(\frac{x}{m^{r'}}, \beta') = 1$  so if  $r' \geq r$ ,  $(\frac{x}{m^{r'}}, \beta\beta') = 1$  and hence, if  $c = \beta\beta' m^{r'}$ ,  $(x, c) = m^{r'}$ . But  $c$  is a common multiple of  $b$  and  $b'$  and  $N_{x,c} \in \mathcal{B}$ . Hence  $\mathcal{B}$  is a base.

**THEOREM 2.1.** *If the shift maps  $n \rightarrow n \pm 1$  are continuous and  $\mathcal{G}$  divisor complete, then  $\mathcal{G}$  is flat.*

*Proof.* If  $b \in G_a$  then  $f^{-1}(N_{a,b})$  is open so there is a  $b'$  with  $b \mid b'$  such that  $b' \in G_{a+1}$ . But this implies  $b \in G_{a+1}$  so  $G_a \subset G_{a+1}$ . Using the left shift we obtain the reverse implication, so  $G_a = G_{a+1}$ . Since this holds for all  $a$ ,  $\mathcal{G}$  is flat.  $\blacksquare$

If  $m = 1$  so  $G_a = \{b \geq 1 : (a, b) = 1\}$  we generate the so-called coprime topology  $\tau_1$ . Note that in this case, each  $\mathcal{G}$  is divisor complete.

**THEOREM 2.2.** *The space  $(\mathbb{Z} - \{0\}, \tau_1)$  is  $T_2$ , second countable, non-compact space, with no isolated points.*

*Proof.* The topology is  $T_2$ : If  $x < y$  let  $p$  be a prime number with  $y - x < p$ . Then  $N_{x,p} \cap N_{y,p} = \emptyset$  since, if not,  $y - x = np$  for some integer  $n$ , which is impossible. It is second countable, being a first countable topology on a countable set. Since, for all primes  $p$ :

$$\mathbb{Z} \setminus N_{0,p} = \bigcup_{a=1}^{p-1} N_{a,p}$$

and each  $(a, p) = 1$  when  $1 \leq a \leq p - 1$ , each  $N_{0,p}$  is closed in the coprime topology. If  $(p_i)$  is an enumeration of all primes,  $(N_{0,p_i})$  has the finite intersection property, with empty intersection. Hence  $\tau_1$  is not compact. There are no isolated points since the topology is weaker than the full topology.  $\blacksquare$

Note that  $G_0 = \{1\}$  and  $N_{0,1} = \mathbb{Z}$ , so  $(\mathbb{Z}, \tau_1)$  fails to be  $T_1$ .

EXAMPLE 2.1. In the coprime topology the set of prime integers is dense, i.e.  $\overline{\mathbb{P}} = \mathbb{Z}$ . This follows directly from Dirichlet's theorem [1]. Indeed the result  $\overline{\mathbb{P}} = \mathbb{Z}$  in  $\tau_1$  is equivalent to Dirichlet's theorem.

THEOREM 2.3. *The space  $(\mathbb{Z}, \tau_G)$  is  $T_1$ , first countable and makes  $\mathbb{Z}$  a topological ring in which the usual operations are continuous. It is also metrizable and has  $\text{Ind}(\mathbb{Z}) = 0$ .*

*Proof.* 1.  $\tau_G$  is  $T_1$ : Given  $x$  and  $y$  in  $\mathbb{Z}$  with  $x \neq y$  let  $b$  be an element of  $G_x = G$  with  $b > x - y$ . (Such an element exists because of the assumption  $G \neq \{1\}$ .) Then  $y \notin N_{x,b}$ .

2.  $\mathbb{Z}$  is  $1^\circ$ :  $(N_{a,b} : b \in G)$  is a countable base for the neighbourhoods of  $a$ .

3.  $\mathbb{Z}$  is a topological ring: This follows directly because, for all  $x$  and  $y \in \mathbb{Z}$  and  $b \in G$ :

$$N_{x,b} + N_{y,b} \subset N_{x+y,b} \text{ and } N_{x,b} \cdot N_{y,b} \subset N_{xy,b}.$$

4. Since  $\mathbb{Z}$  is countable and first countable, it is second countable. Since, for every  $b \in G$

$$\mathbb{Z} = \bigcup_{0 \leq a < b} N_{a,b}$$

and the union is disjoint, each  $N_{a,b}$  is closed as well as open. Therefore the topology has small inductive dimension zero and is  $T_2$ . Therefore [3]  $\text{Ind}(\mathbb{Z}) = 0$ , so the space is also normal. Hence by Urysohn's metrization theorem, the topology is metrizable. ■

THEOREM 2.4. *The space  $(\mathbb{Z}, \tau_G)$  is homeomorphic to  $\mathbb{Q}$ .*

*Proof.* By the theorem of Sierpinski [9], the rationals are characterized topologically by the properties metric, countable and having no isolated points. The only property to prove is the last, which follows immediately because every non-empty open subset of  $\tau_G$  contains a set  $N_{a,b}$  so is infinite. ■

### 3. COMPLETIONS

A non-archimedean quasi-valuation on a ring  $R$  is a function  $v : R \rightarrow [0, \infty)$  such that, for all  $a$  and  $b$  in  $R$ :

- (1)  $v(0) = 0$ ,
- (2)  $v(a) > 0$  for  $a \neq 0$ ,
- (3)  $v(a + b) \leq \max\{v(a), v(b)\}$ ,
- (4)  $v(ab) \leq \min\{v(a), v(b)\}$ .

A pseudo-valuation has (4) replaced by  $v(ab) \leq \max\{v(a), v(b)\}$  and a valuation by  $v(ab) = v(a)v(b)$ , [5].

Below we will refer to a non-archimedean quasi-valuation as simply a valuation.

Note that if  $a \mid b$  then  $v(b) \leq v(a)$  and that for each strictly positive real number  $\delta$ ,  $\{x \in R \mid v(x) \leq \delta\}$  is a closed ideal in  $R$  in the topology induced by  $v$ .

Construction of the completion of a ring with a quasi-valuation proceeds in the normal manner, [5].

Let  $G$  be a semigroup. Define a particular quasi-valuation on  $\mathbb{Z}$  as follows: let  $1 = n_0 < n_1 < n_2$  be a strictly increasing sequence of elements of  $G$  with  $n_1 \mid n_2 \mid n_3 \cdots$  and such that for all  $i \in G$  there is a  $j$  such that  $i \mid n_j$ . For example,  $G = \mathbb{N}$ ,  $n_i = i!$ .

If  $a = 0$  let  $v(a) = 0$ . Otherwise let  $\langle a \rangle = \max\{n_i : n_i \mid a\}$  and then set  $v(a) = 1/\langle a \rangle$ .

**THEOREM 3.1.** *The function  $v$  is a non-archimedean quasi-valuation on  $\mathbb{Z}$  such that the associated metric  $d(x, y) = v(x - y)$  generates the topology  $\tau_G$ .*

*Proof.* Since both topologies are homogeneous we need only consider neighbourhoods of 0. Because  $B(0, 1/n_j] = n_j\mathbb{Z}$ , each  $B(0, 1/n_j]$  is open in  $\tau$ . Conversely, given  $i \in G$  there is a  $j$  with  $i \mid n_j \mid n_{j+1}$  and therefore

$$B(0, \frac{1}{n_j}) = N_{0, n_{j+1}} \subset N_{0, i}.$$

■

**DEFINITION 3.1.** Let  $G$  be a semigroup. We say a sequence  $(x_n)$  of integers is  $G$ -Cauchy if for all  $i \in G$  there is an  $N_i$  such that for all  $n, m \geq N_i$ ,  $i \mid x_n - x_m$ . By Cauchy we mean  $\mathbb{N}$ -Cauchy.

DEFINITION 3.2. If  $(x_n)$  and  $(y_n)$  are two  $G$ -Cauchy sequences we say they are equivalent if for all  $i \in G$  there is an  $N_i$  such that for all  $n \geq N_i$ ,  $i \mid x_n - y_n$ .

DEFINITION 3.3. If  $(x_n)$  is a sequence of integers and  $x_o$  an integer we say  $(x_n)$  converges to  $x_o$  if for all  $i \in G$  there is an  $N_i$  such that for all  $n \geq N_i$ ,  $i \mid x_n - x_o$ . When this is so we write  $x_n \rightarrow x_o$ .

EXAMPLE 3.1. Let  $(\alpha_i)$  be any sequence of integers and for each  $n \in \mathbb{N}$  let  $x_n = \sum_{j=1}^n j! \alpha_j$ . Then  $(x_n)$  is Cauchy.

DEFINITION 3.4. If  $G$  is a semigroup,  $\mathbb{Z}^G$  is the completion of  $\mathbb{Z}$  with respect to the valuation  $v_G$ .

THEOREM 3.2. The ring  $\mathbb{Z}^G$  can be identified with (1) the set of equivalence classes of  $G$ -Cauchy sequences, with (2) the completion of  $(\mathbb{Z}, \tau_G)$  as a topological ring, with (3) the inverse limit

$$\mathbb{Z}^G \approx \lim_{\leftarrow, b \in G} \mathbb{Z}/b\mathbb{Z},$$

and, if  $G$  is divisor complete, with (4)

$$\mathbb{Z}^G \approx \prod_{p \in G} \mathbb{Z}_p$$

where the product is of rings of  $p$ -adic integers, one for each rational prime  $p$  in  $G$ .

*Proof.* (1) Let  $(x_n)$  be  $G$ -Cauchy. Given  $b = n_i \in G$  there is an  $N_b \in \mathbb{N}$  such that  $b \mid x_n - x_m$  for all  $n, m \geq N_b$ . Then  $v(b) \geq v(x_n - x_m)$ . But  $\langle b \rangle = n_i$  which can be made arbitrarily large. Hence  $(x_n)$  is  $v$ -Cauchy.

Conversely, given  $N \in \mathbb{N}$ , let  $(x_n)$  be such that  $v(x_n - x_m) < 1/N$  for all sufficiently large  $n, m$ . Then  $\langle x_n - x_m \rangle \geq N$  so there is an  $n_i$  with  $n_i \geq N$  and  $n_i \mid x_n - x_m$ . The result now follows because given  $b \in G$  we can choose  $N$  so  $b \mid n_i$ .

(2) This follows directly from (1) since  $v_G$  induces the topology  $\tau_G$  on  $\mathbb{Z}$ .

(3) If  $(x_n)$  is a  $G$ -Cauchy sequence in  $\mathbb{Z}$  and  $b \in G$  is given, then for all  $n, m$  sufficiently large,  $x_n \equiv x_m \pmod{b}$ . So each sequence maps to a well

defined class in  $\mathbb{Z}/b\mathbb{Z}$ . It is easy to see that this map is independent of the representative for each element of the completion  $\mathbb{Z}^G$  and that these maps commute with the natural surjections  $\mathbb{Z}/b\mathbb{Z} \rightarrow \mathbb{Z}/c\mathbb{Z}$  when  $b \mid c$ . Hence the completion may be identified with the inverse limit

$$\mathbb{Z}^G \approx \lim_{\leftarrow, b \in G} \mathbb{Z}/b\mathbb{Z},$$

with the ordering for the limit being induced by divisibility.

(4) The given inverse limit can be specified with a compatible system of residue representatives  $(x_b)$ , i.e. such that  $x_m \equiv x_n \pmod{m}$  whenever  $m \mid n$ . To specify an element of  $\mathbb{Z}_p$ , a similar compatible system  $(x_{p^n})$  must be given. If  $G$  is divisor complete, then all prime powers appear and the identification follows as an application of the ring theoretic version of the Chinese Remainder Theorem. ■

**COROLLARY 3.1.** *The space  $\mathbb{Z}^G$ , is homeomorphic to the Cantor set  $\{0, 1\}^{\aleph_0}$ .*

*Proof.* Since  $v_G$  gives rise to a non-archimedean metric, the completion also is non-archimedean, so is totally disconnected. It is also totally bounded since  $\mathbb{Z}$  is a dense totally bounded subset. Hence the completion is metric and compact. It is also infinite and has no isolated point, since the same is true of  $\mathbb{Z}$ . These properties characterise the Cantor set, see e.g. [8]. ■

**THEOREM 3.3.** *The completion  $\mathbb{Z}^G$  has no nonzero nilpotent elements. Each element  $b \in G$  is a non-zero-divisor in  $\mathbb{Z}^G$ . If  $G$  is divisor dense, then  $\mathbb{Z}^G$  has characteristic zero.*

*Proof.*

1.  $\hat{\mathbb{Z}}$  has no non-zero nilpotent elements  $x$ : Let  $x^m = 0$  where  $x = [(x_n)]$ . Then for all  $i \in G$  there is an  $N_i$  such that  $i^m \mid x_n^m$  for all  $n \geq N_i$ , since  $G$  is a semigroup. Hence  $i \mid x_n$  for all  $n \geq N_i$  and therefore  $x = 0$ .

2. Let  $b \in G$  and let  $a$  be an element of  $\hat{\mathbb{Z}}$  such that  $b \cdot a = 0$ . Let  $a = [(a_i)]$  where  $(a_i)$  is Cauchy. Then  $ba_i \rightarrow 0$  so for all  $i \in G$  there is an  $N_i$  such that  $i \mid ba_j$  for all  $j \geq N_i$ . Applying this to  $bi$  implies  $i \mid a_j$  and hence  $a_j \rightarrow 0$  so therefore  $a = 0$ . Hence  $b$  is a non-zero-divisor.

3.  $\mathbb{Z}^G$  has characteristic zero: let  $p \in \mathbb{P}$  and let  $a$  be an element of  $\hat{\mathbb{Z}}$  such that  $p \cdot a = 0$ . Since  $G$  is divisor complete there is a  $b \in G$  such that  $p \mid b$ . Then  $b \cdot a = 0$  so  $a = 0$ . ■

The following is a concrete realization of the result [4] for zero dimensional compact rings:

**THEOREM 3.4.** *Let  $G$  be a semigroup. There exists a family  $(I_n : n \in G)$  of ideals in  $\mathbb{Z}^G$ , consisting of sets which are both open and closed (hence compact), satisfies  $a \mid b \implies I_b \subset I_a$  for all  $a, b \in G$ , and is a basis of neighborhoods of 0 in  $\mathbb{Z}^G$ .*

*Proof.* If  $b \in G$  let  $I_b = \overline{N_{0,b}}$ , where the closure is taken in  $\mathbb{Z}^G$  and  $N_{a,b}$  is the same doubly infinite arithmetic progression in  $\mathbb{Z}$  defined above. Since  $N_{0,b}$  is an ideal, so is  $I_b$ .

Claim:  $\mathbb{Z}^G = \cup_{a=1}^b \overline{N_{0,b}}$  where the union is disjoint. To see this note firstly that  $\cup_{a=1}^b \overline{N_{0,b}} = \overline{\mathbb{Z}} = \mathbb{Z}^G$ . If  $x \in \overline{N_{a,b}} \cap \overline{N_{a',b}}$ , then there are Cauchy sequences  $(x_n)$  and  $(x'_n)$  such that  $a + x_n b \rightarrow x$  and  $a' + x'_n b \rightarrow x$ . But this means that for all  $i \in G$ ,  $i \mid a + x_n b - x$  and  $i \mid a' + x'_n b - x$  for all  $n \geq N_i$  so  $i \mid a - a' + (x_n - x'_n)b$ . Choosing  $i = b$  we get  $b \mid a - a'$  so  $a = a'$ . Therefore the union is disjoint. This implies each  $I_b$  is open as well as closed.

We can write  $I_b = b\mathbb{Z}^G$  (since if  $(bx_n)$  is Cauchy so is  $(x_n)$ ), and therefore  $a \mid b$  implies  $I_b \subset I_a$ .

Finally note that, for each  $b \in G$ ,  $I_b = \{x \in \mathbb{Z}^G : v(x) \leq 1/n_j\}$  where  $n_j$  is the integer appearing in the definition of  $v$  with  $n_j = \max\{n_i : n_i \mid b\}$ , so the  $(I_b)$  are a basis of neighborhoods of 0 and generate the topology on  $\mathbb{Z}^G$ . ■

**THEOREM 3.5.** *Let  $p \in G$  be a rational prime. Then  $p$  is prime in  $\mathbb{Z}^G$ .*

*Proof.* Let  $p = xy$  where  $x = [(x_n)]$  and  $y = [(y_n)]$  where  $(x_n)$  and  $(y_n)$  are Cauchy. Since  $p \mid x_n y_n - p$  for all  $n \geq N_1$ ,  $p \mid x_n y_n$  so  $p \mid x_n$  or  $p \mid y_n$ , and thus either  $p \mid x_n$  or  $p \mid y_n$  for an infinite number of integers  $n \in \mathbb{N}$ .

Suppose  $p \mid x_n$  for an infinite number of  $n \in \mathbb{N}$ . Then, since  $(x_n)$  is Cauchy,  $p \mid x_n$  for all  $n \geq N_2$ . Let  $pz_n = x_n$  for these  $n$ . Then  $(z_n)$  is Cauchy, and if we let  $z = [(z_n)]$ ,  $p = pzy$  in  $\mathbb{Z}^G$ . Therefore  $ip \mid pz_n y_n - p$  for all  $i \in G$  with  $i \geq N_3$ , so  $i \mid z_n y_n - 1$  and hence  $1 = zy$  in  $\mathbb{Z}^G$ . Therefore  $x = pz$  where  $z$  is a unit, so  $x$  is prime in  $\mathbb{Z}^G$ . ■

**EXAMPLE 3.2.** Let  $p \in G$  be a rational prime and  $I = \overline{N_{0,p}} = p\hat{\mathbb{Z}}$  be the principal ideal generated by  $p$ . It follows of course from the above theorem that  $I$  is maximal. However we illustrate these ideas with a direct proof: Let  $M$  be an ideal such that  $I \subset M$  and  $x \in M \setminus I$ . Since  $x \notin I$ ,  $p \nmid x$  so  $x = [(x_n)]$  where  $(x_n)$  can be chosen such that  $p \nmid x_n$  for all  $n$ . For each  $n \in \mathbb{N}$ , let  $y_n^1$  and  $y_n^2$  be integers satisfying  $y_n^1 x_n + p y_n^2 = 1$ . Since  $\mathbb{Z}^G$  is sequentially compact, there exists a subsequence  $(n_j)$  of  $\mathbb{N}$  such that  $y_{n_j}^1 \rightarrow y^1, y_{n_j}^2 \rightarrow y^2$ , and  $x_{n_j} \rightarrow x$  in  $\hat{\mathbb{Z}}$ . It follows that  $yx + py = 1$  so  $1 \in M$ . Hence  $I$  is maximal.

A completion which has been of interest is when  $G = \mathbb{N}$ . Denote  $\mathbb{Z}^{\mathbb{N}}$  by  $\hat{\mathbb{Z}}$ , the so called Prufer ring. There is a lot yet to be discovered about  $\hat{\mathbb{Z}}$ . For example it would be nice to be able to derive the Jacobson radical  $\text{Rad}(\hat{\mathbb{Z}})$ , the intersection of the maximal ideals of  $\hat{\mathbb{Z}}$ . The most natural conjecture is that it is  $\{0\}$  of course. Also to generate examples of primes which are not rational or determine whether  $\hat{\mathbb{Z}}$  is Noetherian (or satisfies the ascending chain condition for closed ideals even). The structure of the character group, however, is immediately accessible:

Let  $T = S^1$  be the unit circle with the usual topology and the structure of a topological group via angle addition. If  $H$  is a topological group let

$$\Gamma = H^* = \{\gamma : \gamma : H \rightarrow T \text{ is a continuous homomorphism}\}$$

be the character group of  $G$  endowed with the so-called  $k$ -topology generated by the sets

$$P(K, V_\epsilon) = \{\gamma : \gamma(K) \subset V_\epsilon\}$$

for each compact subset  $K \subset H$  and open interval  $V_\epsilon$  of angle length  $\epsilon$  which is a subset of  $T$ . Then, by for example [6], if  $H$  is compact and metrizable, then  $\Gamma$  is a countable abelian topological group with torsion, which is discrete in the  $k$ -topology. Applying this to the additive structure of  $\hat{\mathbb{Z}}$  we obtain the following:

Define  $T_{\mathbb{Q}} = \{e^{2\pi i q} : q \in \mathbb{Q}\}$  with the discrete topology.

**THEOREM 3.6.** *The dual of  $\hat{\mathbb{Z}}$  is isomorphic to  $T_{\mathbb{Q}}$ .*

*Proof.* 1. Let  $\gamma : \mathbb{Z} \rightarrow T$  be a character. Then  $\gamma$  is uniformly continuous so has a unique extension to a mapping  $\hat{\gamma} : \hat{\mathbb{Z}} \rightarrow T$ . Conversely, given such a  $\hat{\gamma}$ , its restriction to  $\mathbb{Z}$  is a character on  $\mathbb{Z}$ .

2. For each  $q \in [0, 1) \cap \mathbb{Q}$  define the character  $\gamma_q$  by

$$\gamma_q(n) = e^{2\pi i q n}.$$

Then each  $\gamma_q$  is a character on  $\mathbb{Z}$ . To see this we need only demonstrate continuity: if  $x_n \rightarrow x$  in  $(\mathbb{Z}, \tau)$  and  $q = u/v$  with integers  $u$  and  $v$ , then there is an  $N$  such that  $v \mid x_n - x$  for all  $n \geq N$  so

$$\frac{\gamma_q(x_n)}{\gamma_q(x)} = e^{2\pi i q(x_n - x)} = 1$$

and therefore  $\gamma_q(x_n) = \gamma_q(x)$ .

3. Every character on  $\mathbb{Z}$  is of this form: to see this let  $\gamma$  be a character with  $\gamma(1) = e^{2\pi i \alpha}$  with  $\alpha$  irrational. Since the character is  $\tau$  continuous,

there is a  $b \geq 1$  such that

$$\gamma(1 + b\mathbb{Z}) \subset \{e^{2\pi i\theta} : \alpha - \frac{1}{4} < \theta < \alpha + \frac{1}{4}\}.$$

But, since  $\alpha$  is irrational, the left hand set is dense in  $T$ . This contradiction shows that  $\alpha$  must be rational.

4. By 1,2 and 3 every character  $\hat{\gamma} : \hat{\mathbb{Z}} \rightarrow T$  satisfies  $\hat{\gamma}(1) = e^{2\pi i q}$  for some  $q \in \mathbb{Q}$ . Hence the mapping  $\hat{\gamma} \rightarrow \hat{\gamma}(1)$  identifies the character group with the set of points on the circle with angles a rational multiple of  $2\pi$ . This identification is both injective and surjective. To complete the proof we need only show that the  $k$ -topology on  $\hat{\mathbb{Z}}$  is discrete. But this is immediate since  $P(\hat{\mathbb{Z}}, V_\epsilon) = \{\gamma_1\}$  where  $V_\epsilon$  is the symmetric angle neighborhood of 1 on the unit circle of length  $\pi/2$  and  $\gamma_1(x) = 1$  for all  $x \in \hat{\mathbb{Z}}$ . ■

#### 4. CLOSED SUBSETS AND MAPPINGS FOR $(\mathbb{Z}, \tau)$

In this section the topology is always the full topology  $\tau$ .

**THEOREM 4.1.** *For  $k = 2, 3, \dots$  let  $S_k = \{n^k : n = 0, 1, \dots\}$ . Then each  $S_k$  is perfect in  $(\mathbb{Z}, \tau)$ .*

*Proof.* Let  $a \in \overline{S_k} \setminus S_k$ . Then  $a^2 \geq 1$  and  $N_{a,a^2} \cap S_k \neq \emptyset$ . Hence there exist integers  $x$  and  $y$  such that

$$a + a^2x = y^k.$$

and therefore  $a(1+ax) = y^k$ . But  $(a, 1+ax) = 1$ , so each power of a prime dividing  $a$  is a multiple of  $k$ . Hence  $a = \pm b^k$  for some integer  $b \geq 1$ . If  $k$  is odd,  $a = b^k$  or  $a = (-b)^k$  which is a contradiction, so  $S_k$  is closed in this case. If  $k$  is even,  $a$  is not a  $k$ 'th power so  $a = -b^k = -b^{2l}$  say. But  $a$  is not a square, so  $a = -b^{2l}$ .

Let  $p \in \mathbb{P}$  be such that  $p \equiv 3 \pmod{4}$ , and is larger than any prime divisor of  $b$ . Since  $N_{a,p} \cap S_k \neq \emptyset$ . There exist integers  $x$  and  $y$  such that

$$(y^l)^2 + (b^l)^2 = px.$$

This is a contradiction, since  $u^2 + v^2 = m$ ,  $(u, v) = 1$  is possible if and only if all prime divisors  $p$  of  $m$  satisfy  $p \equiv 1 \pmod{4}$ . Hence  $N_{a,a^2} \cap S_k = \emptyset$  so  $S_k$  is closed.

To show  $S_k$  is perfect, observe that since  $n! \rightarrow 0$ ,  $(n! + a)^k \rightarrow a^k$  so every  $k$ 'th power is a limit of distinct  $k$ 'th powers. Hence  $S_k$  has no isolated points. ■

Note that the result does not of course apply when  $k = 1$ : the set  $S_1 = \{0, 1, 2, 3, \dots\}$  is dense in  $\mathbb{Z}$ .

THEOREM 4.2. *The closure of the set of prime integers in  $(\mathbb{Z}, \tau)$  is*

$$\overline{\mathbb{P}} = \mathbb{P} \cup \{-1, 1\}.$$

*Proof.* If  $x \in \mathbb{Z} \setminus \{-1, 0, 1\}$  then  $N_{x, 2|x|} \cap \mathbb{P}$  is  $\{x\}$  if  $x \in \mathbb{P}$  and  $\emptyset$  otherwise, since  $x + 2n|x| = x(1 \pm 2n)$ . Thus  $\mathbb{P}$  includes none of its cluster points and no point in the complement of  $\mathbb{P} \cup \{-1, 0, 1\}$  is in  $\overline{\mathbb{P}}$ . By Dirichlet's theorem, for all  $b \geq 1$ ,  $N_{\pm 1, b} \cap \mathbb{P} \neq \emptyset$  so  $\pm 1 \in \overline{\mathbb{P}}$ . Finally  $0 \notin \overline{\mathbb{P}}$  since  $N_{0, 4} \cap \mathbb{P} = \emptyset$ . ■

THEOREM 4.3. *If  $k \geq 1$  let  $\mathbb{P}_k$  be the set of integers with absolute value having exactly  $k$  prime factors (including multiplicity). Let  $\mathbb{P}_o = \{-1, 1\}$ . Then, for all  $k \geq 0$ :*

- (a)  $\overline{\mathbb{P}_o} \cup \dots \cup \overline{\mathbb{P}_k} = \overline{\mathbb{P}_k}$ ,
- (b)  $\overline{\mathbb{P}_k} \cup \overline{\mathbb{P}_{k+1}} = \overline{\mathbb{P}_{k+1}}$ ,

where the unions in each case are disjoint.

*Proof.* First two observations. For each  $a \in \mathbb{Z}$ :

(1) for all  $b \geq 1$  there is a  $c \in N_{a, b}$  with the number of prime factors  $\Omega(c) = \Omega(a) + 1$ , namely  $c = a + 2nab$ , where  $n$  has been chosen so that  $1 + 2nb$  is prime,

(2) if  $\Omega(a) = k$  then all elements  $c$  of  $N_{a, 2|a|}$  have  $\Omega(c) \geq k$ .

(a) By (1),  $\mathbb{P}_k \subset \overline{\mathbb{P}_{k+1}}$  so the left hand side is a subset of the right hand side. By (2), for  $j > k$ ,  $\overline{\mathbb{P}_{k+1}} \cap \mathbb{P}_j = \emptyset$ . Since

$$\mathbb{Z} = \bigcup_{k=0}^{\infty} \mathbb{P}_k \cup \{0\}$$

it follows that

$$\overline{\mathbb{P}_{k+1}} \subset \overline{\mathbb{P}_o} \cup \dots \cup \overline{\mathbb{P}_{k+1}} \cup \{0\}.$$

But 0 is not in  $\overline{\mathbb{P}_{k+1}}$  since every element  $c$  of  $N_{0, b} \setminus \{0\}$  has  $\Omega(c) \geq \Omega(b)$ , and  $\Omega(b)$  can be made arbitrarily large. Hence the right hand side is a subset of the left hand side and (a) follows.

(b) This is really just a restatement of (a). ■

Another way to express (a): the set of integers with less than or equal to  $k$  prime factors is closed in  $(\mathbb{Z}, \tau)$ .

THEOREM 4.4. *For all  $a, b \in \mathbb{Z}$  the maps  $x \rightarrow ax + b$  are closed and open for  $(\mathbb{Z}, \tau)$ .*

*Proof.* Since the maps  $x \rightarrow -x$  and  $x \rightarrow x + b$  are homeomorphisms, we need only show that for  $a \geq 1$  the map  $x \rightarrow ax$  is closed.

Let  $F \subset \mathbb{Z}$  be closed and let  $x_n \in F$  be such that  $ax_n \rightarrow \alpha$  in  $\tau$ . Then for all  $i \geq 1$  there is an  $N_i \geq 1$  such that for all  $n \geq N_i$ ,  $i \mid ax_n - \alpha$ . Choose  $i = a$  to show  $a \mid \alpha$ . Let  $\alpha = a\beta$  so  $i \mid a(x_n - \beta)$ . Now choose  $i = aj$  to see that  $j \mid x_n - \beta$  so  $x_n \rightarrow \beta$ . Hence  $\beta \in F$  so the mapping is closed.

For all non-zero  $a$ ,  $aN_{r,s} = N_{ar,|a|s}$  so the maps are open also.  $\blacksquare$

**THEOREM 4.5.** *Let  $p \in \mathbb{Z}[x]$  be a polynomial. Then  $p : (\mathbb{Z}, d) \rightarrow (\mathbb{Z}, d)$  is uniformly continuous.*

*Proof.* If  $n_i \mid x - y$  then  $n_i \mid p(x) - p(y)$  so  $\langle p(x) - p(y) \rangle \geq x - y$ .  $\blacksquare$

The potential domain of application of this result is clear: first show it is true for multinomials  $p : \mathbb{Z}[x_1, \dots, x_n] \rightarrow \mathbb{Z}$ . Use uniform continuity to extend each multinomial to a continuous mapping  $\hat{p} : \hat{\mathbb{Z}}[x_1, \dots, x_n] \rightarrow \hat{\mathbb{Z}}$ , so the set  $F = \hat{p}^{-1}\{0\}$  is a compact subset of  $\hat{\mathbb{Z}}^n$ . Then use compactness of study properties of  $F$ , for example its size.

## 5. EXAMPLES

If  $F$  is a compact subset of  $\mathbb{P}$ , then  $F$  is a finite set. This is because compact subsets are closed and have no cluster points.

First note that a special case of Dirichlet's theorem, usually proved using cyclotomic polynomials, follows from the relationship  $\overline{\mathbb{P}} = \mathbb{P} \cup \{-1, 1\}$ . Namely this implies there exist an infinite number of primes in every arithmetic progression of the form  $an + 1$  and  $an - 1$  for every  $a \geq 1$ . To see this consider the case  $an + 1$ . Since  $N_{1,a} \cap \mathbb{P} \neq \emptyset$  there is a prime  $p_1$  and integer  $n_1$  such that  $p_1 = an_1 + 1$ . The result now follows inductively, first replacing  $N_{1,a}$  by  $N_{1,a} \setminus \{p_1\}$  etc.

Now let the set of primes be divided into two disjoint subsets,  $\mathbb{P} = A \sqcup B$ . Let  $\langle A \rangle$  represent the symmetric multiplicative semigroup in  $\mathbb{Z}$  generated by  $A$ , i.e.

$$\langle A \rangle = \{\pm p_1^{\alpha_1} \cdots p_m^{\alpha_m} : m \in \mathbb{N}, \alpha_i \geq 0, p_i \in A\}.$$

**THEOREM 5.1.** *The interior of  $\langle A \rangle$  is empty in  $(\mathbb{Z}, \tau)$  if and only if the number of primes in  $B$  is infinite.*

*Proof.* Let  $|B| < \infty$  so  $F = \cup_{p \in B}$  is closed in  $(\mathbb{Z}, \tau)$ . If  $P = \mathbb{Z} \setminus F$  then  $P$  is open and non-empty, because if  $q \notin B$  is prime, then  $q \in P$ . If  $n \in P$

$$n = \pm \prod p_i^{\alpha_i}$$

where no  $p_i \in B$ . Hence  $n \in \langle A \rangle$ . Therefore  $P \subset \langle A \rangle$  so the interior is not empty.

Now let  $\langle A \rangle \neq \emptyset$  so some  $N_{a,b} \subset \langle A \rangle$ . Then  $N_{a,b} = (a,b)(\alpha + \beta\mathbb{Z})$  where  $(\alpha, \beta) = 1$ . But given  $p \in \mathbb{P}$ ,  $p \mid \beta$  implies that for all  $n \in \mathbb{Z}$ ,  $p \nmid \alpha + n\beta$ , and  $p \nmid \beta$  implies there is an  $n \in \mathbb{Z}$  such that  $p \mid \alpha + n\beta$ . Hence the only primes which can be missing from  $A$  are among prime divisors of  $\beta$ , which are finite in number. Hence  $B$  has a finite number of elements. ■

We say a subset  $A$  of a topological space  $X$  is discrete if all points of  $A$  are isolated in  $X$ . In the theorem below, the metric  $d$  is the same as that defined in Theorem 3.1 above.

**THEOREM 5.2.** *Let  $A$  be a non-empty subset of  $\mathbb{Z}$  with  $0 \notin A$  and  $(a, b) = 1$  if  $a, b \in A$  with  $a \neq b$ . If  $A$  is complete in  $(\mathbb{Z}, d)$ , then  $A$  is finite.*

*Proof.* For each  $a \in A$ ,  $N_{a,2|a|} \cap A = \{a\}$ , so the derived set  $A' = \emptyset$  and  $A$  is discrete. Embed  $A$  in the completion  $\hat{\mathbb{Z}}$ . Then  $A$  is closed hence compact in the completion, hence sequentially compact in the completion, therefore in  $\mathbb{Z}$ , so it is compact in  $\mathbb{Z}$ . Since  $A$  is discrete and compact it must be finite. ■

**EXAMPLE 5.1.** For  $n = 0, 1, \dots$  let  $f_n = 2^{2^n} + 1$  so  $\mathbb{F} = \{f_n\}$  is the Fermat numbers. Then  $\mathbb{F}$  is closed and discrete in  $(\mathbb{Z}, \tau)$ : Let  $f_{n_i} \rightarrow \alpha \neq 0$  with  $n_1 < n_2 < \dots$ . Since  $|\alpha| \mid f_{n_i} - \alpha$  for  $n_i$  sufficiently large,  $\alpha \mid f_{n_i}$ . Therefore  $\alpha \mid (f_{n_i}, f_{n_{i+1}}) = 1$  so  $\alpha = \pm 1$ . But  $1 \notin \mathbb{F}$  since  $3 \mid 2^{2^j}$  is false.

Also  $0 \notin \mathbb{F}$  since  $N_{0,2} \cap \mathbb{F} = \emptyset$ , and finally,  $-1 \notin \mathbb{F}$  since  $4 \nmid 2^{2^{n_i}} + 1 - (-1)$ . Hence  $\mathbb{F}$  is closed in  $(\mathbb{Z}, \tau)$ . It is discrete since  $(f_n, f_m) = 1$  for  $n \neq m$ .

**EXAMPLE 5.2.** Let  $\mathbb{M} = \{m_p = 2^p - 1 : p \in \mathbb{P}\}$  be the Mersenne numbers. Then  $\mathbb{M}$  is closed and discrete in  $(\mathbb{Z}, \tau)$ .

We assume the following well known property of divisors of the Mersenne number  $m_p$ : If  $n \mid m_p$  then  $n \equiv \pm 1 \pmod{8}$  and  $n \equiv 1 \pmod{p}$ , see for example [7].

Firstly  $0 \notin \mathbb{M}$  since  $m_p \in N_{0,2} \cap \mathbb{M}$  implies  $2 \mid m_p$  and  $2 \not\equiv \pm 1 \pmod{8}$ .

If  $\alpha \neq 0$  is such that  $m_{p_i} \rightarrow \alpha$  then  $\alpha \mid m_{p_i}$  so  $\alpha \equiv 1 \pmod{p_i}$ . But we can chose  $p_1 < p_2 < \dots$  and in particular such that  $|\alpha| < p_i$ , so necessarily  $\alpha = 1$ . If then  $m_p \in \{N_{1,4} \setminus m_2\} \cap \mathbb{M}$ , there is an integer  $n$  such that  $2^p - 1 = 1 + 4n$  which is impossible for  $p > 2$ . Hence  $\mathbb{M}$  is closed.

Let  $p_1 < p_2 < \dots < p_n$  be all the primes up to  $p_n$  and suppose  $m_{p_n}$  is a cluster point of  $\mathbb{M}$ . Then

$$\mathbb{M} \cap N_{m_{p_n}, 2^{p+1}} \setminus \{2^{p_1} - 1, \dots, 2^{p_n} - 1\} \neq \emptyset.$$

Therefore there is a prime  $q > p_n$  with  $2^p - 1 + n2^{p+1} = 2^q - 1$  so  $2^q = 2^p(2n + 1)$  which is impossible. Therefore  $\mathbb{M}$  is discrete.

EXAMPLE 5.3. Let  $Sf = \{\pm n : n \geq 2 \text{ squarefree}\}$ . Then  $Sf$  is perfect in  $(\mathbb{Z}, \tau)$ :  $Sf$  is closed since if the squarefree sequence  $n_i \rightarrow \alpha$  then  $\alpha \mid n_i$  so  $\alpha$  is squarefree. Every point of  $Sf$  is a cluster point because if  $a$  is squarefree and  $b \geq 1$  then  $N_{a,b} \setminus \{a\}$  contains a square free element: write  $a + nb = (a, b)(\alpha + n\beta)$  where  $(\alpha, \beta) = 1$ . Since the arithmetic progression  $\alpha + n\beta$  has an infinite number of prime values, chose one which does not divide  $(a, b)$  so that the corresponding  $a + nb$  will be square free and distinct from  $a$ .

EXAMPLE 5.4. Let  $U = \{u_n : n = 0, 1, 2, \dots\}$  be the Fibonacci numbers where  $u_0 = 0, u_1 = 1$  and  $u_{n+2} = u_{n+1} + u_n$  for all  $n \geq 0$ . Then

1. The point 0 is a cluster point of  $U$ , i.e.  $0 \in U'$ . This is because for all  $b \geq 1$ , there is an  $n$  such that  $b \mid u_n$ .

2. Every point congruent to 4 mod 8 is not in  $\overline{U}$ : If for some  $n$  and  $l$ ,  $u_n = 4 + 8l$ , then  $4 \mid u_n$  so  $6 \mid n$  which implies  $8 = u_6 \mid u_n$ , so  $8 \mid 4$  which is impossible. This can be written  $U \cap N_{4,8} = \emptyset$ .

3. Similarly it may be shown that  $U \cap N_{6,12} = \emptyset$  and also that  $U \cap N_{7,21} = \emptyset$ .

4. More generally, it follows from the proposition below, that if  $(i, j) = 1$  or 2 then there is a  $b \geq 1$  such that  $a = u_i u_j$  implies  $U \cap N_{a,b} = \emptyset$ .

5. It follows from 2,3 and 4 above that the following points of  $\mathbb{Z} \setminus U$  are not in  $\overline{U}$ :  $\{-6, -4, 4, 6, 7, 10, 12\}$ .

6. **Conjecture:** The closure of  $U$  in  $(\mathbb{Z}, \tau)$  is  $U \cup V$  where  $V = \{(-1)^{n+1} u_n : n \in \mathbb{N}\}$ .

Strong numerical evidence for the truth of this conjecture comes from consideration of the union of the basic open sets  $N_{a,b}$  with the following given values of  $a$  and  $b$ , all being such that their intersection with  $U$  is empty:

$a \in \{4, 6\}$ ,  $b = 8$ ,  
 $a \in \{4, 6, 7, 9\}$ ,  $b = 11$ ,  
 $a \in \{6\}$ ,  $b = 12$ ,  
 $a \in \{4, 6, 7, 9\}$ ,  $b = 13$ ,  
 $a \in \{4, 6, 10, 12, 14\}$ ,  $b = 16$ ,  
 $a \in \{6, 7, 10, 11\}$ ,  $b = 17$ ,  
 $a \in \{4, 6, 7, 9, 11, 12, 14\}$ ,  $b = 18$ ,  
 $a \in \{4, 6, 7, 9, 10, 12, 14\}$ ,  $b = 19$ ,  
 $a \in \{4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 19\}$ ,  $b = 21$ ,  
 $a \in \{4, 7, 16, 19\}$ ,  $b = 23$ ,  
 $a \in \{4, 6, 9, 11, 12, 14, 15, 18, 19, 20, 22\}$ ,  $b = 24$ ,  
 $a \in \{4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 27\}$ ,  $b = 29$

To check these, simply consider the values of the Fibonacci numbers modulo  $b$  for one complete period.

PROPOSITION 5.1. (a) Let  $(i, j) = 1$ ,  $i, j \geq 3$ . Then

$$u_n = u_i u_j + u_i u_j u_{ij}^l$$

has no solution in integers  $n \geq 3$  and  $l \in \mathbb{Z}$ .

(a) If  $(i, j) = 2$ ,  $i, j \geq 3$ . Then

$$u_n = u_i u_j + u_i u_j u_{ij/2}^l$$

has no solution.

*Proof.* (a) Let  $n \geq 3$ ,  $i, j \geq 3$ ,  $(i, j) = 1$ , and  $l \in \mathbb{Z}$  be such that

$$(1) \quad u_n = u_i u_j + u_i u_j u_{ij}^l$$

Then  $u_i \mid u_n$  and  $u_j \mid u_n$  and therefore  $i \mid n$  and  $j \mid n$  and therefore  $ij \mid n$  so  $u_{ij} \mid u_n$ . Equation (1) then implies  $u_{ij} \mid u_i u_j$ . But  $(u_i, u_j) = u_{(i,j)} = u_1 = 1$ , so  $u_{ij} = u_i u_j$ .

If  $\alpha = \frac{1+\sqrt{5}}{2}$  then we can write  $u_n = \left[ \frac{\alpha^n}{\sqrt{5}} + \frac{1}{2} \right]$  therefore

$$\frac{\alpha^{ij}}{\sqrt{5}} - \frac{1}{2} \leq \left( \frac{\alpha^i}{\sqrt{5}} + \frac{1}{2} \right) \left( \frac{\alpha^j}{\sqrt{5}} + \frac{1}{2} \right)$$

so

$$(2) \quad \alpha^{ij} \leq \frac{\alpha^{i+j}}{\sqrt{5}} + \frac{\alpha^i + \alpha^j}{2} + \frac{3\sqrt{5}}{4}.$$

But  $1 < \alpha$  and therefore  $\alpha^{ij} \leq c\alpha^{i+j}$  where  $c = \frac{1}{\sqrt{5}} + \frac{1}{2} + \frac{3\sqrt{5}}{4}$ . Therefore

$$ij \leq \frac{\log c}{\log \alpha} + i + j.$$

where  $2 < i, j$ . This equation has no solutions, hence neither does (1).

(b) Let  $n \geq 3$ ,  $l \in \mathbb{Z}$  and  $i, j \geq 4$  be such that  $(i, j) = 2$ . Let

$$(3) \quad u_n = u_i u_j + u_i u_j u_{ij/2} l.$$

Using a similar argument to that given in (a) it follows that

$$(4) \quad u_{ij/2} = u_i u_j.$$

Let  $i = 2r$  and  $j = 2s$  so  $u_{2r} u_{2s} = u_{2rs}$ . Therefore

$$\frac{\alpha^{2rs}}{\sqrt{5}} - \frac{1}{2} \leq \left( \frac{\alpha^{2r}}{\sqrt{5}} + \frac{1}{2} \right) \left( \frac{\alpha^{2s}}{\sqrt{5}} + \frac{1}{2} \right).$$

Using the same argument as that given in (a), but replacing  $\alpha$  by  $\alpha^2$ , we obtain the inequality

$$rs \leq \frac{\log c}{2 \log \alpha} + r + s$$

where  $r, s \geq 2$ , so  $rs < 2 + r + s$ . The only solution to this inequality is  $(r, s) = (2, 3)$  or  $(3, 2)$ . In that case

$$u_i u_j = u_4 u_6 = 24 \neq 144 = u_{12}$$

so equation (4) is never true. Therefore (3) has no solutions.  $\blacksquare$

### ACKNOWLEDGMENT

The support of the Department of Mathematics at the University of Waikato, and discussions with Ian Hawthorn and Tom Körner are warmly acknowledged, as are suggestions made by John Cremona, especially those relating to the use of the inverse limit to describe the completion.

### REFERENCES

1. Apostol, T.M. *Introduction to Analytic Number Theory*. New York, Berlin Heidelberg: Springer-Verlag, 1976.
2. Aigner, M. and Ziegler, G. M, *Proofs from the Book*, Springer-Verlag, 1998.
3. Engelking, R. *Outline of General Topology*, North-Holland, 1968.
4. Kaplansky, I. *Topological Rings*, Amer. J. Math. **69** (1947), 153-183.
5. Mahler, K. *P-adic numbers and their functions.*, Cambridge, Cambridge University Press, 1981
6. Morris, S. A. *Pontryagin Duality and the structure of locally compact abelian groups* London Mathematical Society Lecture Notes in Mathematics **29**, Cambridge, 1977.
7. Reibenboim, P. *The New Book of Prime Number Records*, Springer-Verlag, 1999.
8. Schoenfeld, A. H. and Gruenhage, G. *An alternate characterization of the Cantor set*, Proc. Amer. Math. Soc. **53**, (1975), 235-236.
9. Sierpinski, W. *Sur une propriété topologique des ensembles dénombrables eenses en soi*, Fundam. Math. **1**, (1920), 11-16.