

Examples for the paper: On shifted primes and balanced primes

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Abstract

These graded examples illustrate the proof techniques in the paper.

1 Single shift examples

In the following set of examples we show how one might extend the range of the shifted decomposition by relaxing the constraint $a \mid A(\mathbf{e})$.

Example 1: Here we count primes p such that $p + 1 = 2 \cdot 3^2 u$ where u is squarefree and coprime to 2 and 3. We impose the additional constraint that $5 \nmid u$. The leading term of an asymptotic expansion for the number of such primes up to x is $c_1 \cdot Li(x)$ and the aim of the example is to derive $c_1 = 0.039\dots$

In deriving the leading term we have the summation

$$\Sigma_1 = \sum_{\substack{p:p \leq x \\ p+1=18n \\ 5 \nmid n, 2 \nmid n, 3 \nmid n}} \sum_{d:d^2 \mid n} \mu(d)$$

We can assume that p is odd. The condition $5 \nmid (p + 1)/18$ is equivalent to $p + 1 \equiv 18, 55, 72 \pmod{90}$ where “,” represents “or”, and where the option

36 is omitted because the only prime solution would be $p = 5$. Similarly $3 \nmid (p + 1)/18$ is equivalent to $p + 1 \equiv 18 \pmod{54}$. The condition $2 \nmid u$ is represented by restricting d to be odd. Therefore we can write

$$\Sigma_1 = \sum_{\substack{d \geq 1 \\ d \text{ odd}}} \mu(d) \sum_{\substack{p \leq x: \\ p+1 \equiv 0 \pmod{18d^2} \\ p+1 \equiv 18, 54, 72 \pmod{90} \\ p+1 \equiv 18 \pmod{54}}} 1.$$

By Lemma ?? there are conditions for the joint congruences in the inner sum to have any solution. For example $(90, 18d^2) \mid 18, 54, 72$ which is the same as $(5, d^2) \mid 1$, so we derive the condition $5 \nmid d$. Each choice of an optional congruence gives rise, asymptotically, to the same number of primes, giving the leading coefficient 3. Hence

$$\begin{aligned} \frac{\Sigma_1}{Li(x)} &= 3 \cdot \sum_{d \geq 1, 2, 3, 5 \nmid d} \frac{\mu(d)}{\phi(\{18d^2, 90, 54\})} \\ &= \frac{1}{24} \cdot \prod_{p \neq 2, 3, 5} \left(1 - \frac{72}{\phi(\{18p^2, 90, 54\})} \right) \\ &= \frac{1}{24} \cdot \prod_{p > 5} \left(1 - \frac{1}{p^2 - p} \right) \\ &= 0.039.. \end{aligned}$$

Example 2: In this example the form is $p + 1 = u$ with u squarefree and

not divisible by 3 or 5. We derive the constant $c_2 = 0.177\dots$

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p \leq x \\ p+1=u, u \text{ squarefree} \\ 3 \nmid u, 5 \nmid u}} 1 \\
&= \sum_{\substack{p:p \leq x \\ p+1 \equiv 2 \pmod 3 \\ p+1 \equiv 2,3,4 \pmod 5}} \sum_{d:d^2|p+1} \mu(d) \\
&= \sum_{\substack{d:1 \leq d \\ 3 \nmid d, 5 \nmid d}} \mu(d) \sum_{\substack{p:p \leq x \\ p+1 \equiv 0 \pmod{d^2} \\ p+1 \equiv 2 \pmod 3 \\ p+1 \equiv 2,3,4 \pmod 5}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 3 \cdot \sum_{\substack{1 \leq d \\ 3 \nmid d, 5 \nmid d}} \frac{\mu(d)}{\phi(\{d^2, 15\})} \\
&= \frac{3}{8} \prod_{p \neq 3,5} \left(1 - \frac{1}{p^2 - p}\right) \\
&= 0.177\dots
\end{aligned}$$

Example 3: In this example, as in Example 1, the form is $p + 1 = 18u$ with u squarefree and not divisible by 2,3 or 5. We derive the constant $c_3 = 0.039\dots$. In this case we enforce the requirements $2 \nmid u$ and $3 \nmid u$ by making $6u$ squarefree. Since $5 \nmid u$ we must have $u \equiv 1, 2, 3, 4 \pmod 5$ so $p + 1 = 18u \equiv 18, 36, 54, 72 \pmod{90}$. But the option $p + 1 \equiv 35 \pmod{90}$ has

no solutions, so is omitted. Again we apply Lemma ?? to show $5 \nmid d$:

$$\begin{aligned}
\Sigma_1 &= \sum_{\substack{p:p \leq x \\ p+1=3(6u), 6u \text{ squarefree} \\ 5 \nmid u}} 1 \\
&= \sum_{\substack{p \leq x \\ p+1 \equiv 18, 54, 72 \pmod{90}}} \sum_{d:3d^3 | p+1} \mu(d) \\
&= \sum_{1 \leq d} \mu(d) \sum_{\substack{p:p \leq x \\ p+1 \equiv 0 \pmod{3d^2} \\ p+1 \equiv 18, 54, 72 \pmod{90}}} 1 \\
\frac{\Sigma_1}{Li(x)} &\sim 3 \cdot \sum_{\substack{d \geq 1 \\ 5 \nmid d}} \frac{\mu(d)}{\phi(\{3d^2, 90\})} \\
&= \frac{1}{8} \prod_{p \neq 5} \left(1 - \frac{24}{\phi(\{3p^2, 90\})} \right) \\
&= 0.039..
\end{aligned}$$

Example 4: In this example the form is $p + 7 = 18u$ with u squarefree and not divisible by 2,3 or 5. We derive the constant $c_4 = c_3 = 0.039...$ We observe that the constant is the same for all prime shifts $k \geq 7$. The only difference between this and Example 3 is in the step where we consider $p + 7 \equiv 18, 36, 54, 72 \pmod{90}$ the value 72 is discarded rather than 36.

Example 5: In this example the form is $p + 5 = 18u$ with u squarefree and again not divisible by 2,3 or 5. We derive the constant $c_5 = 4c_4/3$. As for Example 4, the only difference between this and Example 3 is the step $p+5 \equiv 18, 36, 54, 72 \pmod{90}$ wherein all residues contribute, so the multiplier is 4 rather than 3.

Finally note that if $2 \mid k$ or $3 \mid k$ there are no primes of the form $p + k = 18u$ except when $k = 15$, wherein the only prime is 3, and $k = 16$ wherein the only prime is 2.

2 Double shift examples

We now proceed to count primes p where $p+k = a \cdot u, p-k = b \cdot v$ and where in each case $a, b = p_1^{e_1} \cdots p_l^{e_l}$ for some $e_i \geq 0$, u is squarefree and coprime to kab , and as before, k, a, b are pairwise coprime. Many combinations of the e_i 's do not give rise to an infinite set of primes, so initially the situation seems quite complicated:

Example 7: Let $k = a = 1$ and $B = \{2, 3\}$ so $b = 6$. If $p+k = 2^{e_1}3^{e_2} \cdot u$ and $p-k = 2^{f_1}3^{f_2} \cdot v$, and $p > 3$ then necessarily one of e_2 or f_2 is zero and the other greater than or equal to 1, e_1 or $f_1 = 1$ and the other is strictly greater than 1. To illustrate consider a situation which meets these constraints where we assume the prime p is odd and u, v squarefree:

$$\begin{aligned} p+1 &= 2^3 \cdot u, \\ p-1 &= 2 \cdot 3^2 \cdot v. \end{aligned}$$

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{p:p \leq x \\ p+1=8u \\ p-1=18v \\ u, v \text{ squarefree}}} 1 \\ &= \sum_{\substack{p:p \leq x \\ p+1 \equiv 8 \pmod{16} \\ p+1 \equiv 8, 16 \pmod{24} \\ p-1 \equiv 18 \pmod{36} \\ p-1 \equiv 18, 36 \pmod{54}}} \sum_{\substack{a, b: 8a^2 | p+1 \\ 18b^2 | p-1}} \mu(a)\mu(b) \\ &= \sum_{\substack{d: 1 \leq d \\ 2, 3 \nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p \leq x \\ p \equiv w \pmod{72d^2} \\ p+1 \equiv 8 \pmod{16} \\ p+1 \equiv 8 \pmod{24} \\ p-1 \equiv 18 \pmod{36} \\ p-1 \equiv 18, 36 \pmod{54}}} 1 \\ \frac{\Sigma_1}{Li(x)} &\sim 2 \cdot \sum_{\substack{d \geq 1 \\ 2, 3 \nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{72d^2, 16, 24, 36, 54\})} \\ &= \frac{2}{144} \prod_{p \neq 2, 3} \left(1 - \frac{2}{p^2 - p} \right) \\ &= 0.011.. \end{aligned}$$

Note that in the second step inner sum the two given conditions imply that $(a, b) = 1$ and the residue w is determined by the Chinese Remainder Theorem. In the third step the choice $p + 1 \equiv 16 \pmod{24}$ has been omitted, since it gives rise to at most $O(1)$ primes.

Example 8: Keep the same requirements as in the previous example but add the constraint $(u, 5) = 1$, $(v, 5) = 1$. This is reflected in two new equations

$$\begin{aligned} p + 1 &\equiv 8, 16, 24, 32 \pmod{40} \\ p - 1 &\equiv 18, 36, 54, 72 \pmod{90} \end{aligned}$$

where the residues 16 and 54 can be omitted apriori. Then use Lemma ?? to prune out the sets of congruences which don't have any common solution. This reduces the number of sets of congruences arising from line 3 below from 18 to 4, accounting for the numerator of the leading coefficient.

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{p:p \leq x \\ p+1=8u \\ p-1=18v \\ u, v \text{ squarefree}}} 1 \\ &= \sum_{\substack{p:p \leq x \\ p+1 \equiv 8 \pmod{16}, p+1 \equiv 8, 16 \pmod{24} \\ p+1 \equiv 8, 16, 24, 32 \pmod{40} \\ p-1 \equiv 18 \pmod{36}, p-1 \equiv 18, 36 \pmod{54} \\ p-1 \equiv 18, 36, 54, 72 \pmod{90}}} \sum_{\substack{a, b: 8a^2 | p+1 \\ 18b^2 | p-1}} \mu(a)\mu(b) \\ &= \sum_{\substack{d: 1 \leq d \\ 2, 3, 5 \nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p:p \leq x \\ p \equiv w \pmod{72d^2} \\ p+1 \equiv 8 \pmod{16} \\ p+1 \equiv 8 \pmod{24} \\ p+1 \equiv 8, 24, 32 \pmod{40} \\ p-1 \equiv 18 \pmod{36} \\ p-1 \equiv 18, 36 \pmod{54} \\ p-1 \equiv 18, 36, 72 \pmod{90}}} 1 \\ \frac{\Sigma_1}{Li(x)} &\sim 4 \cdot \sum_{\substack{d \geq 1 \\ 2, 3, 5 \nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{72d^2, 16, 24, 40, 36, 54, 90\})} \\ &= \frac{4}{576} \prod_{p \neq 2, 3, 5} \left(1 - \frac{2}{p^2 - p} \right) \\ &= 0.0062.. \end{aligned}$$

Example 9: Let $k = 2$ and $B = \{3, 5\}$ so $b = 15$. If $p + 2 = 2^{e_1} 3^{e_2} \cdot u$ and $p - 2 = 2^{f_1} 3^{f_2} \cdot v$, and $p > 5$ then necessarily at most one of e_1 and f_1 is greater than zero and at most one of e_2 and f_2 is greater than zero. 8.1 First we count odd primes p where there exist u, v squarefree and:

$$\begin{aligned} p + 2 &= 3^3 \cdot u, \\ p - 2 &= 5^2 \cdot v. \end{aligned}$$

with $(u, 2.3.5) = (v, 2.3.5) = 1$ so necessarily $(u, v) = 1$. Then

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{p: 7 \leq p \leq x \\ p+2=27u \\ p-2=25v \\ u, v \text{ squarefree}}} 1 \\ &= \sum_{\substack{p: p \leq x \\ p+2 \equiv 27 \pmod{54} \\ p+2 \equiv 27, 54 \pmod{81} \\ p+2 \equiv 27, 54, 81, 108 \pmod{135} \\ p-2 \equiv 25 \pmod{50} \\ p-2 \equiv 25, 50 \pmod{75} \\ p-2 \equiv 25, 50, 75, 100 \pmod{125}}} \sum_{\substack{a, b: 27a^2 | p+2 \\ 25b^2 | p-2}} \mu(a)\mu(b) \\ &= \sum_{\substack{d: 1 \leq d \\ 2, 3, 5 \nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p: p \leq x \\ p \equiv w \pmod{675d^2} \\ p+2 \equiv 27 \pmod{54} \\ p+2 \equiv 27, 54 \pmod{81} \\ p+2 \equiv 54 \pmod{135} \\ p-2 \equiv 25 \pmod{50} \\ p-2 \equiv 50 \pmod{75} \\ p-2 \equiv 25, 50, 75, 100 \pmod{125}}} 1 \\ \frac{\Sigma_1}{Li(x)} &\sim 8 \cdot \sum_{\substack{d \geq 1 \\ 2, 3, 5 \nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{675d^2, 54, 81, 135, 50, 75, 125\})} \\ &= \frac{8}{5400} \prod_{p \neq 2, 3, 5} \left(1 - \frac{2}{p^2 - p}\right) \\ &= 0.0013.. \end{aligned}$$

Note that in the second step inner sum the two given conditions imply that $(a, b) = 1$ and the residue w is determined by the Chinese Remainder

Theorem. In the third step the congruences $p+2 \equiv 27 \pmod{135}$ and $p-2 \equiv 25 \pmod{75}$ have been omitted, since they give rise to at most $O(1)$ primes. The congruences $p-2 \equiv 81, 108 \pmod{135}$ have also been omitted because when checked against the last set of congruences they fail condition of Lemma ??.

Example 10: Now we generalize Example 8. Let $k = 2$ and $P = \{3, 5\}$ so $\wp = 15$.

10.1 First we compute Σ_1 which is the number of odd primes p where there exist u, v squarefree and $p+2 = 3^l \cdot u$, $p-2 = 5^m \cdot v$ for some $l, m \geq 1$ and $(u, 2.3.5) = (v, 2.3.5) = 1$. Then

$$\begin{aligned}
\Sigma_1(l, m) &= \sum_{\substack{p: 7 \leq p \leq x \\ p+2=3^l \cdot u \\ p-2=5^m \cdot v \\ u, v \text{ squarefree}}} 1 \\
&= \sum_{\substack{p: p \leq x \\ p+2 \equiv 3^l \pmod{2 \cdot 3^l} \\ p+2 \equiv 3^l, 2 \cdot 3^l \pmod{3^{l+1}} \\ p+2 \equiv 3^l, 2 \cdot 3^l, 3 \cdot 3^l, 4 \cdot 3^l \pmod{5 \cdot 3^l} \\ p-2 \equiv 5^m \pmod{2 \cdot 5^m} \\ p-2 \equiv 5^m, 2 \cdot 5^m \pmod{3 \cdot 5^m} \\ p-2 \equiv 5^m, 2 \cdot 5^m, 3 \cdot 5^m, 4 \cdot 5^m \pmod{5 \cdot 5^m}}} \sum_{\substack{a, b: 3^l \cdot a^2 | p+2 \\ 5^m \cdot b^2 | p-2}} \mu(a)\mu(b) \\
&= \sum_{\substack{d: 1 \leq d \\ 2, 3, 5 \nmid d}} \tau^*(d)\mu(d) \sum_{\substack{p: p \leq x \\ p \equiv w \pmod{3^l \cdot 5^m \cdot d^2} \\ p+2 \equiv 3^l \pmod{2 \cdot 3^l} \\ p+2 \equiv 3^l, 2 \cdot 3^l \pmod{3 \cdot 3^l} \\ p+2 \equiv 2 \cdot 3^l \pmod{5 \cdot 3^l} \\ p-2 \equiv 5^m \pmod{2 \cdot 5^m} \\ p-2 \equiv 2 \cdot 5^m \pmod{3 \cdot 5^m} \\ p-2 \equiv 5^m, 2 \cdot 5^m, 3 \cdot 5^m, 4 \cdot 5^m \pmod{5 \cdot 5^m}}} 1 \\
\frac{\Sigma_1(l, m)}{Li(x)} &\sim 8 \cdot \sum_{\substack{d \geq 1 \\ 2, 3, 5 \nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{3^l \cdot 5^m \cdot d^2, 2 \cdot 3^l, 3 \cdot 3^l, 5 \cdot 3^l, 2 \cdot 5^m, 3 \cdot 5^m, 5 \cdot 5^m\})} \\
&= \frac{8}{\varphi(2 \cdot 3^{l+1} \cdot 5^{m+1})} \prod_{p \neq 2, 3, 5} \left(1 - \frac{2}{p^2 - p}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\Sigma_1}{Li(x)} &\sim \left(\sum_{l,m \geq 1} \frac{8}{3^l \cdot 5^m (2.4)} \right) \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p} \right) \\
&= \frac{1}{8} \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p} \right) \\
&= 0.112..
\end{aligned}$$

10.2 Now count odd primes p where there exist u, v squarefree and $p+2 = 3^l 5^m \cdot u$, $p-2 = v$ for some $l, m \geq 1$. We spare the reader the details, but in step 3, for fixed l, m , we arrive at 8 sets of congruences, the same number as in 9.1 (but different congruences). Hence $\Sigma_2 = \Sigma_1$

10.3 Next we count odd primes p where there exist u, v squarefree and some $l \geq 1$ so $p+2 = 3^l \cdot u$, $p-2 = v$. Here the pruning of the congruences at step three using Lemma ?? depends on the equivalence class of l modulo 4, but, fortunately, we are left with the same number, 4 of valid classes in each case. The reader is spared the details. This leads to

$$\begin{aligned}
\frac{\Sigma_3(l)}{Li(x)} &\sim 4 \cdot \sum_{\substack{d \geq 1 \\ 2,3,5 \nmid d}} \frac{\tau^*(d)\mu(d)}{\phi(\{3^l \cdot d^2, 2 \cdot 3^l, 3 \cdot 3^l, 5 \cdot 3^l, 2, 3, 5\})} \\
&= \frac{4}{\varphi(2 \cdot 3^{l+1} \cdot 5)} \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p} \right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\Sigma_3}{Li(x)} &\sim \left(\sum_{l \geq 1} \frac{4}{8 \cdot 3^l} \right) \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p} \right) \\
&= \frac{1}{4} \prod_{p \neq 2,3,5} \left(1 - \frac{2}{p^2 - p} \right) \\
&= 0.224..
\end{aligned}$$

10.4 Note that there are *no* odd primes $p > 5$ such that there exist u, v squarefree so $(u, 30) = 1$, $(v, 30) = 1$ and for some $l \geq 1$: $p + 2 = 5^l \cdot u$, $p - 2 = v$, since the second equation gives $p \equiv 1 \pmod{3}$ so by the first $3 \mid u$. Now note also that there are *no* primes $p > 5$ where there exist u, v squarefree so $(u, 30) = 1$, $(v, 30) = 1$ and $p + 2 = u$, $p - 2 = v$.

10.5 Finally we combine the computations in 10.1-10.5 to count the primes $p > 5$ up to x satisfying

$$\begin{aligned} p + 2 &= \alpha \cdot u, \\ p - 2 &= \beta \cdot v \end{aligned}$$

where $P = \{3, 5\}$ and $\alpha, \beta \in \langle P \rangle$ and $(uv, 30) = 1$:

$$\begin{aligned} \frac{\Sigma}{Li(x)} &= \frac{2(2\Sigma_1 + \Sigma_3)}{Li(x)} \\ &= \prod_{p \neq 2, 3, 5} \left(1 - \frac{2}{p^2 - p} \right) \\ &= 0.89.. \end{aligned}$$

where the leading coefficient has simplified to 1! Numerical evaluation of other examples indicates that this formula should hold more generally.