Test 1 (2008) Part B math 310 Solutions

(1) (a) A prime number is a natural number $p > 1$ such that its only divisors are 1 and p. The standard factorization of a natural number $n \geq 1$ is 1 if $n = 1$ and, if $n > 1$

$$n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$$

where the p_i are distinct prime numbers and $\alpha_i \geq 1$ are natural numbers. If $p_1 < p_2 < \dots < p_m$ this factorization is unique.

(b) Let $a - b = 1$ and let $p \mid (a,b) \Rightarrow p \mid a$ and $p \mid b$

$\Rightarrow a = p\alpha,\ b = p\beta$ for some integers $\alpha, \beta \Rightarrow a - b = (\alpha - \beta)p$

$\Rightarrow p \mid 1 \Rightarrow p = 1$. Hence $(a,b) = 1$.

(c) Let the number of primes p_i be finite and let $\{p_1, \dots, p_n\}$ be all of them(??) then $N = p_1 \cdots p_n + 1 > p_i$ for $1 \leq i \leq n$

$\therefore N$ is not prime. Therefore it has a prime divisor p. But $p = p_i$ for some $i \Rightarrow p \mid N - p_1 \cdots p_n = 1$, which is false. $\therefore$ N has a prime divisor different from the $\{p_i\}$, a contradiction (!!).

Hence, the number of primes is infinite.

(d) Bertrand's postulate is that there is at least one prime in $[n, 2n)$ for all $n = 1, 2, \dots$

$[1,2) = \{1\}$ no primes
$[2,4) \ni 3$
$[3,6) \ni 5$
$[4,8) \ni 5$.

2 (a) A function $f : \mathbb{N} \to \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$ even is

<u>multiplicative</u> if $\forall\, a, b$ with $(a,b) = 1$, $\underline{f(ab) = f(a) \cdot f(b)}$.

$f(n) = n^2 \Rightarrow f(ab) = (ab)^2 = a^2 \cdot b^2 = f(a) \cdot f(b)$.

Hence f is (completely) multiplicative.

$f(n) = 2n \Rightarrow f(2 \cdot 3) = 12 \neq f(2) \cdot f(3) = 4 \cdot 6 = 24$

Hence f is <u>not</u> multiplicative.

(b)
$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{if } \exists\, p \in \mathbb{P}\ \ p^2 \mid n \\ (-1)^m & \text{if } n = p_1 \ldots p_m \text{, a product of } m \text{ distinct primes.} \end{cases}$$

Assume ✩ $$\sum_{d \mid n} \mu(d) f(d) = \prod_{p \mid n} (1 - f(p))$$

Then $u(n) = 1\ \forall\, n \in \mathbb{N}$ is multiplicative.

Hence $$\sum_{d \mid n} \mu(d) \cdot 1 = \prod_{p \mid n} (1 - u(p)) = \prod_{p \mid n} 0 = 0$$

since the product is not empty.

(c) $$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right).$$

since the Dirichlet product of multiplicative functions is multiplicative,

$\mu * \mu$ is multiplicative. Hence $(\mu * \mu)(1) = 1$ & if

$n = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$ $\quad (\mu * \mu)(n) = \prod_{j=1}^{m} \mu * \mu(p_j^{\alpha_j})$.

Now $$(\mu * \mu)(p^\alpha) = \sum_{d \mid p^\alpha} \mu(d) \mu(p^\alpha / d) = \sum_{j=0}^{\alpha} \mu(p^j) \mu(p^{\alpha - j})$$

$\alpha = 1 \quad = \mu(1)\mu(p) + \mu(p)\mu(1) = -2$

$\alpha = 2 \quad = \mu(1)\mu(p^2) + \mu(p)\mu(p) + \mu(p^2)\mu(1) = 1$

$\alpha > 2 \quad = \mu(1)\mu(p^\alpha)\ \mu(p)\mu(p^{\alpha-1}) + \mu(p^2)\mu(p^{\alpha-2}) + \ldots$

$= 0$

Hence If $n = p_1 \dots p_m q_1^2 \dots q_\ell^2$ q_i, p_j distinct, prime

$(\mu * \mu)(n) = (-2)^m$

else $(\mu * \mu)(n) = 0$.

(d) $f(n) = (u * g)(n)$ $u(n) = 1 \ \forall n$.

$\Rightarrow$ $(\mu * f)(n) = (\mu * (u * g))(n)$

$= ((\mu * u) * g)(n)$

$= (I * g)(n)$ where $I(n) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$

$= g(n)$.

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3) (a) Can generate all soln using parameters $(p, q) \in \mathbb{N}^2$

via $x = 2pq$ $p > q$

$y = p^2 - q^2$

$z = p^2 + q^2$

Switching x & y we $20 = 2pq \Rightarrow 10 = pq$ so

all. $p = 5$ and $q = 2$ so $y = p^2 - q^2 = 25 - 4 = 21$ &

$z = p^2 + q^2 = 29$

& Pythagorean triple is $(20, 21, 29)$

(b) n can be expressed as the sum of 2 squares $\Leftrightarrow$ every prime p dividing n to an odd power is $p \equiv 1 \bmod 4$. Let

$n = 3^4 5^3 17^1 19^1$. Here $19 \equiv 3 \bmod 4$ so n cannot be written as the sum of 2 squares.

(c) $1 = 1^2 + 0^2 + 0^2$

$2 = 1^2 + 1^2 + 0^2$

$3 = 1^2 + 1^2 + 1^2$

$4 = 2^2 + 0^2 + 0^2$

$5 = 2^2 + 1^2 + 0^2$

$6 = 2^2 + 1^2 + 1^2$

$7 = ?$

$8 = 2^2 + 2^2 + 0^2$

& these are the 1st 7 smallest.

$\therefore$ 7 cannot be written $7 = x^2 + y^2 + z^2$.

(d) By inspection $15 = 3^2 + 2^2 + 1^2 + 1^2$

$$\Rightarrow \quad 101^2 \cdot 15 = 101^2(3^2+2^2+1^2+1^2)$$
$$= (101\times 3)^2 + (101\times 2)^2 + 101^2 + 101^2$$
$$= 303^2 + 202^2 + 101^2 + 101^2.$$

4) (a) $(n|p) = \begin{cases} 0 & p|n \\ 1 & \text{if } x^2 \equiv n \bmod p \text{ has a sol}^n\ x. \\ -1 & \text{if } \ldots\ldots\ldots\ldots \text{ no sol}^n. \end{cases}$

) $0^2 \equiv 0 \bmod 7$, $1^2 \equiv 1 \bmod 7$, $2^2 \equiv 4 \bmod 7$, $3^2 \equiv 2 \bmod 7$

$\therefore$ $\{0,1,2,4\}$ are quadratic residues. The set of non-zero ones is $|\{1,2,4\}| = \frac{p-1}{2} = 3$.

) $(n|p) \equiv n^{\frac{p-1}{2}} \bmod p \Rightarrow (-1|p) \equiv (-1)^{\frac{p-1}{2}} \bmod p$. But LHS $= \pm 1$ RHS $= \pm 1$ are $p > 2 \Rightarrow (-1|p) = (-1)^{\frac{p-1}{2}}$.

$$f(ab) = (ab|p) \equiv (ab)^{\frac{p-1}{2}} \bmod p$$
$$\equiv a^{\frac{p-1}{2}} \cdot b^{\frac{p-1}{2}} \bmod p$$
$$\equiv f(a).f(b) \bmod p$$

$p|a$ or $p|b \Leftrightarrow p|ab$
so $f(a).f(b) = 0 = f(ab)$

t each side is ± 1 $\therefore$
$$f(ab) = f(a).f(b).$$

) $(2|p) = (-1)^{\frac{p^2-1}{8}}$. If p & q are distinct odd primes
$$(p|q).(q|p) = (-1)^{\frac{(p-1)(q-1)}{4}}.$$

e) $$(5.7^2.17|11) = (5|11)(7|11)^2(17|11)$$
$$= (11|5)(-1)^{\frac{(11-1)(5-1)}{4}} . 1 . (6|11)$$
$$= (1|5) . 1 . 1 . (2|11)(3|11)$$
$$= 1^3 . (-1)^{\frac{11^2-1}{8}} (11|3) (-1)^{\frac{(11-1)(3-1)}{4}}$$
$$= (-1)(2|3)(-1) = -1.$$