The general equation

$$y^2 = x^3 + ax^2 + bx + c.$$

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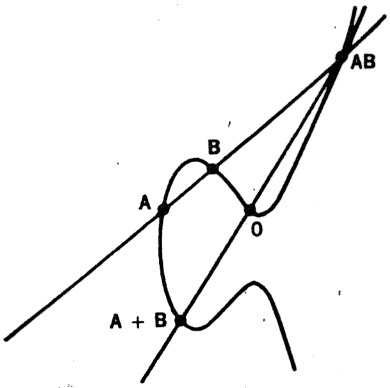
In order that this should define an elliptic curve, it is necessary and sufficient that $D \neq 0$ where

$$D = a^2b^2 - 4a^3c - 4b^3 + 18abc - 27c^2$$



Addition of $A = (-3, 2)$ and $B = (-1, 4)$ on the elliptic curve $y^2 = x^3 - 7x + 10$, using $0 = (1, 2)$ as the zero element. Here $AB = (5, 10)$ and $A + B = (-2, -4)$.

If $x_1 \neq x_2$, then the coordinates of $\mathbf{P}_1 + \mathbf{P}_2$ are

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - a - x_1 - x_2,$$

$$y_3 = -y_1 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_3 - x_1),$$

If $y_1 \neq 0$, then the coordinates of $2\mathbf{P}_1$ are

$$x_3 = \left(\frac{3x_1^2 + 2ax_1 + b}{2y_1}\right)^2 - a - 2x_1,$$
$$y_3 = -y_1 - \left(\frac{3x_1^2 + 2ax_1 + b}{2y_1}\right)(x_3 - x_1). \tag{5.53}$$

Using these formulae, it is a simple matter to calculate the coordinates of the sum of two points. For example, we find that the first 10 multiples of the point $(1, 2)$ on the curve $x^3 - 7x + 10 = y^2$ are as follows. We give the coordinates both as decimals and as fractions.

n	x_n	y_n	x_n	y_n
1	1.00000	2.00000	1/1	2/1
2	−1.00000	−4.00000	−1/1	−4/1
3	9.00000	−26.00000	9/1	−26/1
4	2.25000	2.37500	9/4	19/8
5	−3.16000	−0.75200	−79/25	−94/125
6	2.59763	−3.05690	439/169	−6716/2197
7	6.42112	15.15917	4681/729	298378/19683
8	−1.52891	4.13865	−8831/5776	1816769/438976
9	1.24409	−1.79358	364121/292681	−283996102/158340421
10	239.30450	3701.68885	13215591/55225	48040055236/12977875

[From "An Introduction to the Theory of Numbers" Niven, Zuckerman, Montgomery 1991]

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[1984]

On the Conjecture of Birch and Swinnerton-Dyer for an Elliptic Curve of Rank 3

By Joe P. Buhler, Benedict H. Gross and Don B. Zagier

Abstract. The elliptic curve $y^2 = 4x^3 - 28x + 25$ has rank 3 over Q. Assuming the Weil-Taniyama conjecture for this curve, we show that its *L*-series $L(s)$ has a triple zero at $s = 1$ and compute $\lim_{s \to 1} L(s)/(s-1)^3$ to 28 decimal places; its value agrees with the product of the regulator and real period, in accordance with the Birch-Swinnerton-Dyer conjecture if Ш is trivial.

The object of this note is to verify the conjecture of Birch and Swinnerton-Dyer numerically (to high accuracy) for the elliptic curve

$$E: y^2 = 4x^3 - 28x + 25. \tag{1}$$

Integral points on E

x	y	n_0	n_1	n_2	$\hat{h}(P)$	$h(P)$
−3	0	0	−1	−1	1.50192454	1.09861229
−2	3	0	−1	1	1.36857251	.69314718
−1	3	−1	0	−1	1.20508110	0.00000000
0	2	1	0	0	.99090633	0.00000000
1	0	0	1	0	.66820517	0.00000000
2	0	0	0	1	.76704336	.69314718
3	3	1	1	0	1.18592770	1.09861229
4	6	−1	−1	−1	1.46677848	1.38629436
8	21	1	−1	0	2.13229530	2.07944154
11	35	−1	−1	1	2.43916362	2.39789527
14	51	0	2	0	2.67282066	2.63905733
21	95	0	0	−2	3.06817342	3.04452244
37	224	−2	0	−1	3.62493152	3.61091791
52	374	1	−1	2	3.96137952	3.95124372
93	896	2	2	1	4.53836901	4.53259949
342	6324	−2	0	1	5.83640586	5.83481074
406	8180	0	2	2	6.00769815	6.00635316
816	23309	1	3	−1	6.70508531	6.70441435

$y \to 2y + 1$

$\div 4.$

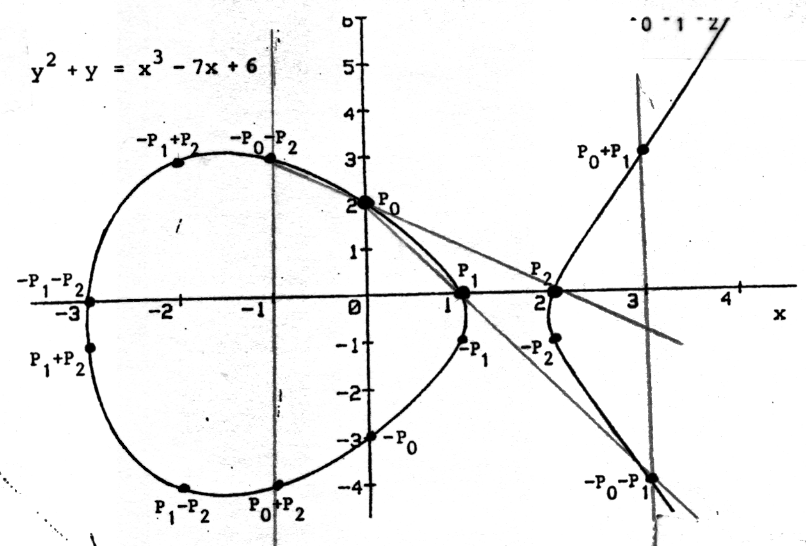

FIGURE 2
Integral points on E