University of Waikato
Department of Mathematics NumberTheory 2005 Assignment 8

Due Friday 10th June 2005: 314B do 1,4,5,6. and 514B 1,2,3,4,7.

1. Define $power(n) = \frac{\log(n)}{\log N(n)}$ where $N(n)$ is the conductor or square free core of n. Prove that $power(n) = 1$ if and only if n is square free and n is powerful implies $power(n) \geq 2$.

2. Prove that the **ABC conjecture** implies there exists only finitely many triples of consecutive powerful numbers.[Hint: apply ABC to $(n^2 - 1) + 1 = n^2$ and note $N(n^2(n^2 - 1)) \leq n^{3/2}$.]

3. Consider the number θ which we derived so that $p(n) = \lfloor \theta^{3^n} \rfloor$ is prime for all n in $\mathbb{N}$. Use Mathematica and the N[*,50] numerical evaluation function to find θ to 50 decimal places. Determine explicitly which primes are values p_n. (E.g. form Table[p[n],PrimeQ[p[n]], n,1,M] and vary the upperbound M until, at this level of approximation for θ, p[n] ceases to be prime and compare what you have with the set of all primes in that range.) Experiment with some known constants of Mathematics to see if it might be related.

4. Let $f(n) = \sin(\frac{\pi(1+(n-1)!}{n})$. Prove that $f(n) = 0$ if and only if n is prime.

5. Let $p_1 = 2$ and for all $n > 0$, and $p_{n+1} = 2^{p_n} - 1$. Use a reasonable amount of computation to verify p_n is prime for as large as possible a range of values of n.

6. Use **Axiom D** to prove that there exist infinitely many quadruples of consecutive integers, each being the product of two distinct primes.

7. Derive an expression for the number of ways change can be made for a note of denomination n in $\mathbb{N}$, given coins of denomination 2 and 3. Check by hand using $n = 4, 5, 6$.

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