University of Waikato
Department of Mathematics
Number Theory Assignment 3

Due Wed 6th April 2005. Normal hand-in instructions.
math54.314B: 1,3,5,6,,8,10; math54.4/514B: 1-10.

1. Verify Euler's theorem $a^{\varphi(m)} \equiv 1 \bmod m$ in case $a = 5, m = 12$. Verify Wilson's theorem in case p=11 and n=10.

2. Show that if $a \mid b$ then $\varphi(a) \mid \varphi(b)$. If $n > 1$ show that the sum of the positive integers less than n and coprime with n is $n\varphi(n)/2$ (Hint: if m satisfies the condition then so does n-m.)

3. Write out a neat proof of that final step in the proof of Euler's summation theorem. Then apply the theorem directly to get a nice asymptotic expression with error for

$$\sum_{n\leq x} \sqrt{n}.$$

4. Apply Euler's summation theorem to get a nice expression for

$$\sum_{n\leq x} \sqrt{n^2+1}.$$

5. Use Mathematica to try out the functions PrimeQ, Prime, Divisors, Length, ListPlot, and Sum. Then define the function $\sigma(n)$. Plot its values and the values of $T(x) = \sum_{n\leq x} \sigma(n)$.

6. Use Mathematica to explore the function $\mu(n)$ (MoebiusMu[n]) through defining the summation function $M(x) = \sum_{1\leq n\leq x} \mu(n)$ and then plotting this (using Plot or ListPlot with PlotJoined $\rightarrow$ True as an additional argument). Would you say $|M(x)| \leq \sqrt{x}$?

7. Use Mathematica to explore the function $\lambda(n)$ (Liouville's function) through defining the summation function $L(x) = \sum_{1\leq n\leq x} \lambda(n)$ and then plotting this function. Would you say that approximately the same proportion of numbers had even and odd "parity"? If not what do you suppose the proportions might be ?

8. Read Andrews Section 15.1 and then do Exercise 1 on p206.

9. Read Stopple Section 4.4 and do the Exercises in this section. Show $d(n) = O(n)$ and $\sigma(n) = O(n^2)$ (best possible is $O(n^\epsilon)$ and $O(n^{1+\epsilon})$ respectively), and comment on how these maximal orders compare with their average values. What are their minimums ?

10. Write out a more careful/comprehensible proof than that in the notes for the inequality $\pi(2^{k+1}) \leq 2^k$ for all $k = 0, 1, 2, \cdots$.

Kevin Broughan
23rd March 2005