University of Waikato
Department of Mathematics
math310-08A: Number Theory Assignment 2

Due Wed 26th March 2008. Normal hand-in instructions.

1. Verify Euler's theorem $a^{\varphi(m)} \equiv 1 \bmod m$ in case $a = 5, m = 12$. Verify Wilson's theorem in case p=11 and n=10.

2. Show that if $a \mid b$ then $\varphi(a) \mid \varphi(b)$.

3. Make up a multiplication and table for the residue class ring $\mathbb{Z}_6$ and give the inverse class for each element which has an inverse (there are $\varphi(6)$ of these).

4. If for $n \geq 0$, F_n is the n'th Fermat number, prove that

$$F_n - 2 = \prod_{j=0}^{n-1} F_j.$$

Hint: use induction.

5. Prove that for all $n \in \mathbb{N}$, $\sigma(n)\phi(n) \leq n^2$. Hint: express each of the terms on the left as products and simplify.

6. Prove that for each prime p in the range $n < p < 2n$, $\binom{2n}{n} \equiv 0 \bmod p$ and$\binom{2n}{n} \not\equiv 0 \bmod p^2$.

7. Show that for each natural number $n > 2$, one of the numbers $n, n+1$ must be composite. Show that, for $n > 3$, one of the numbers $n, n+2, n+4$ must be composite. Find the smallest n such that $n+1, n+2$, and $n+3$ are all composite.

8. Show that the divisor function $d(n)$ (defined as the number of divisors of n) is multiplicative by expressing it as the Dirichlet product of the function which takes the value 1 at every natural number with itself.

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17th March 2008