2 (c) $u = 2x - 2xy \Rightarrow u_x = 2 - 2y$, $u_{xx} = 0$; $u_y = -2x$, $u_{yy} = 0$ $\Rightarrow$ u is harmonic.

Then $\frac{\partial v}{\partial y} = u_x = 2 - 2y \Rightarrow v = 2y - y^2 + f_1(x)$ for some unknown function $f(x) \Longrightarrow$

$2x = -u_y = v_x = f_1'(x)$ so $f_1'(x) = 2x$ & $f_1(x) = x^2 + c$ $c \in \mathbb{R}$.

Hence $v = 2y - y^2 + x^2 + C$ so

$$f(z) = u + iv = 2x - 2xy + i(2y + x^2 - y^2 + c)$$
$$= 2(x+iy) + i(x^2 - y^2 + 2ixy) + ic$$
$$= \underline{2z + iz}$$ where we have taken $C = 0$.

3 (a) If $f(z)$ is analytic inside and on a simple closed curve Γ & $n \geq 0$ then, if a is in the interior of Γ:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^{n+1}} dz.$$

(b) with $n = 1$, (a) $\Rightarrow$ $f'(a) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^2} dz$

Let $|f(z)| \leq M$ $\forall z \in \mathbb{C}$ & Γ be a circle $z = a + re^{i\theta}$

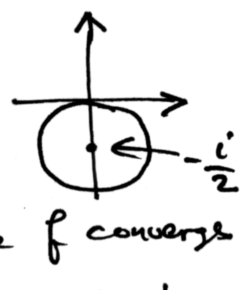
so $|f'(a)| \leq \frac{1}{2\pi} \frac{M(r 2\pi)}{r^2} = \frac{M}{r} \to 0$ as $r \to \infty$

Hence $f'(a) = 0$ $\Rightarrow$, with some work, $f(z) = $ constant on $\mathbb{C}$.

(c) $f(z) = \sum_{n=0}^{\infty} \frac{2^n (z - (-\frac{i}{2}))^n}{(n+2)(n+3)}$ so centre is $\boxed{-\frac{i}{2}}$

& $a_n = \frac{2^n}{(n+2)(n+3)}$ so $\frac{|a_n|}{|a_{n+1}|} = \frac{\cancel{2^n}(\cancel{n+3})(n+4)}{(n+2)(\cancel{n+3})2^{\cancel{n}+1}} \to \frac{1}{2}$

Hence $R_f = \frac{1}{2}$. On the circle $z = -\frac{i}{2} + \frac{1}{2}e^{i\theta}$



so $f(z) = \sum_{n=0}^{\infty} \frac{e^{ni\theta}}{(n+2)(n+3)} = \sum_{n=0}^{\infty} b_n$ &

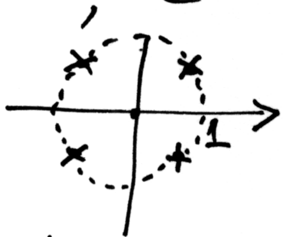
$|b_n| = \frac{1}{(n+2)(n+3)} \leq \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$. Hence f converges at <u>every</u> point on the circle of convergence (where is it <u>not</u> analytic?)

4 (a) $1 + z^4 = 0 \iff z^4 = -1 = e^{i\pi + 2ni\pi} \quad n \in \mathbb{Z}$

$\Rightarrow z = e^{\frac{i\pi}{4} + \frac{ni\pi}{2}} \quad n \in \mathbb{Z} \Rightarrow$ singularities are

$e^{i\pi/4}, \quad e^{i\pi/4 + i\pi/2}, \quad e^{i\pi/4 + \frac{2i\pi}{2}}, \quad e^{i\pi/4 + \frac{3i\pi}{2}}$

or $\dfrac{\pm 1 \pm i}{\sqrt{2}}$



$$f(z) = \frac{1}{1+z^4} = \frac{1}{1-(-z^4)} = 1 + (-z^4) + (-z^4)^2 + \dots (-z^4)^n + \dots$$

$$= 1 - z^4 + z^8 + \dots (-1)^n z^{4n} + \dots$$

& R_f is the distance from 0, the centre, to the closest singularity, when $z^4 = -1 \Rightarrow |z|^4 = 1 \Rightarrow |z| = 1$ so $R_f = 1$.

(b) If f is analytic in the given annulus, it may be represented as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \quad \text{where}$$

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^{n+1}} dz,$$

Γ being a simple closed path traversing the annulus once in an anticlockwise direction.

(c) $f(z) = \dfrac{1}{z^2(4-z)} = \dfrac{1}{4z^2} \cdot \dfrac{1}{1 - \frac{z}{4}} \qquad |z| < 4 \iff \left|\frac{z}{4}\right| < 1$

$$= \frac{1}{4z^2} \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \frac{1}{4z^2}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \dots + \frac{z^n}{4^n}\right]$$

$$= \frac{1}{4z^2} + \frac{1}{16z} + \frac{1}{64} + \frac{z}{4^4} + \dots \frac{z^n}{4^{n+3}} + \dots$$