UNIVERSITY OF WAIKATO
Department of Mathematics

MATH311A - Advanced Calculus

COMPLEX - TEST

Friday 28 May 2004 - 3.00-4.00pm

Time allowed: 50 minutes

Attempt any 3 questions. Each is worth 33 1/3 % of the total.

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1. (a) Classify the singularities in $\mathbb{C}$ of $\dfrac{\sin(z)}{z(z+i)(z+4)^2}$.

   (b) Show that $\left|e^{i\theta}\right|=1$ for all $\theta \in \mathbb{R}$.

   (c) Show that if $z=Re^{i\theta}$ and $R>1$ then

   $$\frac{1}{|z+1|}\leq\frac{1}{R-1}$$

   and find a value of θ for which this is an equality.

   (d) Prove that $\{z\in\mathbb{C}:0<\mathrm{Re}(z)<1\}$ is open and connected.

2. (a) Let $f(x)=u+iv$ be analytic on $\mathbb{C}$. Prove that $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ on $\mathbb{C}$.

   (b) Use the Cauchy-Riemann equations to show $f(z)=e^{-z^2}$ is analytic on $\mathbb{C}$.

   (c) Let $u=2x(1-y)$. Find an analytic function f so that $u=Re\,f(z)$.

3. (a) Write down Cauchy's integral formula for the n'*th* derivative of an analytic function.

   (b) Use (a) to show that the only bounded entire functions are constant.

   (c) Find the centre and circle of convergence of the power series

   $$\sum_{n=0}^{\infty}\frac{(2z+i)^n}{(n+2)(n+3)},$$

   and examine the convergence or divergence of the series at each point in $\mathbb{C}$.

4. (a) Find the singularities of $f(z)=\dfrac{1}{1+z^4}$.

   Expand f in a power series about $z=0$ giving the fist 3 non-zero terms and the general term. What is the radius of convergence?

   (b) State Laurent's theorem for a function $f(z)$ analytic in an annulus $r<|z-a|<R$ where $0\leq r<R$ are real numbers and $a\in\mathbb{C}$. Give an expression for the n'*th* Laurent coefficient c_n.

   (c) Expand $f(z)=\dfrac{1}{z^2(4-z)}$ in a Laurent series with centre $z=0$ in $0<|z|<4$ giving 4 terms and the general term.