UNIVERSITY OF WAIKATO
Department of Mathematics

MATH311A - Advanced Calculus

COMPLEX - TEST
Thursday 8 May 2003 - 11.10am-12.00noon

Time allowed: 50 minutes
Attempt any 3 questions. Each is worth 33 1/3% of the total.
If you are asked to give an example you do not need to prove that it has the given property.

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1. (a) Let $a \in \mathbb{C}$ be a complex number and let $\varepsilon > 0$. Define $B(a,\varepsilon)$ and define an open subset of $\mathbb{C}$. Show that $1+i \in B(2-i,5)$.

   (b) Give an example of a bounded open subset of $\mathbb{C}$ and an unbounded subset which is not open.

   (c) If P and Q are open subsets of $\mathbb{C}$, and $P \cap Q \neq \phi$, prove that $P \cap Q$ is an open subset of $\mathbb{C}$.

   (d) EITHER: using the limit definition, prove that for all $z \in \mathbb{C}$

$$\lim_{n\to\infty} \frac{3n-2z}{n+z} = 3$$

   OR: prove that for all $z \in \mathbb{C}$, $\lim_{n\to\infty} \frac{z^n}{n!} = 0$.

2. (a) State the Cauchy-Riemann equations and explain how they are related to analytic functions on open subsets of $\mathbb{C}$.

   (b) Let $\Omega = \{z \in \mathbb{C} : |z-i| < 1\}$. Give an example of a function which is analytic on Ω and one which has one singular point in Ω, but is otherwise analytic.

   (c) Use the Cauchy-Riemann equations to prove that the level curves of the real and imaginary parts of an analytic function cross at right angles.

   (d) Show that $f(z) = e^{-z^2}$ is analytic on $\mathbb{C}$ using the Cauchy-Riemann equations.

3. (a) Define the complex integral $\int_C f(z)\,dz$ and state Cauchy's Integral Formula.

   (b) Give an example of a function and a curve C in $\mathbb{C}$ such that $\oint_C f(z)\,dz = 0$ and one for which $\oint_C f(z)\,dz \neq 0$.

   (c) Prove Cauchy's Theorem, i.e. $\oint_C f(z)\,dz = 0$ if C is a simple smooth closed curve and f is analytic everywhere inside and on C.

   (d) Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{e^z}{z^2-1}\,dz$$

   if C is the circle $|z| = 2$.

4. (a) Define power series and radius of convergence of a power series.

   (b) Give examples of three power series having radii of convergence 0, 1 and ∞, respectively.

   (c) Prove that if a power series $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ converges at $z = z_0 \neq a$, and $|a-z| < |a-z_0|$, then the series converges absolutely at z, OR prove that if $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ and $0 < R < \infty$, then R is the radius of convergence of the series $f(z)$.

   (d) Find the circle of convergence of

$$\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$$

   and examine the convergence or divergence of the series at each point on this circle.