UNIVERSITY OF WAIKATO
Department of Mathematics

0654.311A - Advanced Calculus

TEST 1 - (Complex)

Wednesday 8 May 2002 - 11-12noon

Time allowed: 50 minutes
Attempt any 3 questions. Each is worth 33 1/3 % of the total.

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1. (a) Sketch the set $P=\{z\in\mathbb{C}:|2z-i|<1\}$ and explain why it is open.

   (b) Give an example of an open subset of $\mathbb{C}$ which is unbounded and connected, but not simply connected.

   (c) Let $C=\{z\in\mathbb{C}:|z+3-4i|=2\}$. Find the minimum and maximum values of $|f(z)|$ on C, where $f(z)=z$. How does this illustrate the maximum modulus principle?

   (d) Show that $\left(1+i\sqrt{3}\right)^6=2^6$ and illustrate geometrically.

2. (a) Define the complex integral $\int_\Gamma f(z)dz$ and use your definition to prove that $\int_\Gamma \frac{2}{z}dz=4\pi i$ where Γ is the unit circle.

   (b) State, but do not prove, each of the following:

   (i) Cauchy's theorem,
   (ii) Cauchy's integral formula,
   (iii) Cauchy's integral formula for the n'th derivative,
   (iv) Cauchy's inequality,
   (v) Liouville's theorem for bounded entire functions,
   (vi) The fundamental theorem of algebra.

   (c) Indicate how the theorems in (b) are related by describing the implications between them.

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3. (a) Illustrate geometrically the nature of complex multiplication.

   (b) Illustrate geometrically the nature of complex differentiation.

   (c) Find the image of the sector $S=\{z\in\mathbb{C}:z=re^{i\theta},0\le r\le 2,0\le\theta\le\pi/4\}$ under the mapping $f(z)=z^3$.

   (d) Explain what it means to say $f(z)=\sqrt{z}$ is a "multiple valued function" and what you would do to ensure, in any application, that it was well defined with unique values. Do the same for $g(z)=\sqrt{z^2+1}$.

4. Consider the power series

$$f(z)=\sum_{n=0}^{\infty}\frac{n}{n^3+1}\frac{\left(z-(i+1)\right)^n}{2^n}.$$

   (a) Find the center, radius of convergence, disk of convergence and circle of convergence.

   (b) Examine its convergence or divergence at each point on the circle of convergence.

   (c) Does it converge at $z=i$ and at $z=0$?

   (d) Find a series for $f'(z)$ the derivative. Find a point on its circle of convergence at which it does not converge.