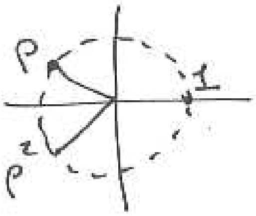
math311 — Advanced Calculus 2007 Test 2 — pg 1

(1) (a) $\rho = e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\Rightarrow \operatorname{Re}\rho = -\frac{1}{2} \quad \operatorname{Im}\rho = \frac{\sqrt{3}}{2}$



$\rho^2 = e^{4\pi i/3} = e^{(\pi i + \pi i/3)}$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$1+\rho+\rho^2 = 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$

(b) $p(z) = z^3 + 1 = (z+1)(z^2 + az + b)$

equating coef. of the constant term gives $1 = 1.b \Rightarrow b = 1$

equating the coef of z^2 gives $0 = 1 + a \Rightarrow a = -1$.

Hence $z^3 + 1 = (z+1)(z^2 - z + 1)$

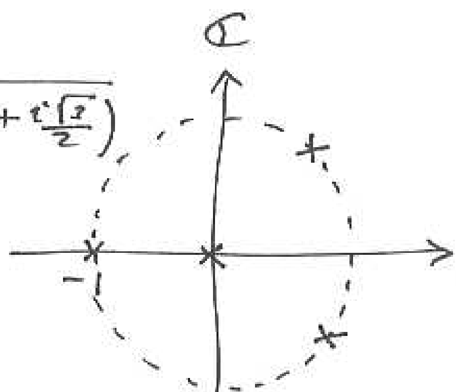
for $z^2 - z + 1$ the roots are $\alpha, \beta = \frac{1 \pm \sqrt{1^2 - 4.1}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

Here $p(z) = (z+1)(z-\alpha)(z-\beta)$

$= (z+1)\left(z - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ //

The Fundamental Theorem of Algebra states that any polynomial with coefficients in $\mathbb{C}$ factors completely into linear factors over $\mathbb{C}$. Here we have a complete set of 3 linear factors for $p(z)$.

(c) $f(z) = \dfrac{1}{z^2(z+1)\left(z-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(z-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$



$z = 0$ is a double pole

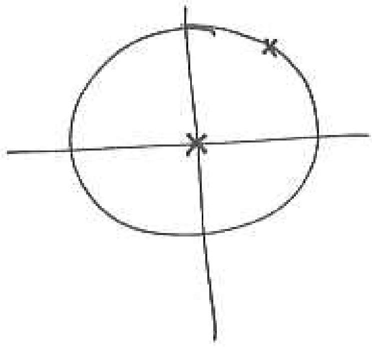
$z = -1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}$ are simple poles.

(d) $f(z) = \dfrac{1}{z^2(1-(-z^3))}$ $\quad r = -z^3$

$= \dfrac{1}{z^2}\left[1 - z^3 + z^6 - \dots\right]$

Laurent Series: $= \dfrac{1}{z^2} - z + z^4 - \dots$

annulus is $\{z : 0<|z|<1\}$.



(2) (a) $f(z) = u + iv$

Cauchy–Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (C-R)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f(z) = |z|^2 = x^2+y^2 \Rightarrow u = x^2+y^2,\ v = 0$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial v}{\partial y} = 0$

$\frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial x} = 0$

So (C-R) $\Rightarrow$ $x=0$ and $y=0$. Thus C-R does not hold on any open subset of $\mathbb{C}$ $\therefore$ $f(z)$ is <u>never</u> holomorphic.

(b) $f(z) = (x+iy)^2 + 4(x+iy) + 3$

$= x^2-y^2+2ixy + 4x + 4iy + 3$

$= (x^2-y^2+4x+3) + i(2xy+4y) = u+iv$

$\Rightarrow$ $u = x^2-y^2+4x+3 \Rightarrow u_x = 2x+4 \quad u_y = -2y$

$v = 2xy+4y \qquad v_x = 2y \quad v_y = 2x+4$

$\Rightarrow$ $u_x = v_y$ and $u_y = -v_x$ on $\mathbb{C}$ $\therefore$ $f(z)$ is holomorphic.

(c) $f'(z) = u_x + iv_x$

$= (2x+4) + i(2y) = 2(x+iy) + 4$

$= 2z+4.$

(d) $f(z) = z^2+4z+3$

$= [(z-i)^2 + 2iz + 1] + 4z + 3$

$= (z-i)^2 + z(2i+4) + 4$

$= (z-i)^2 + (z-i)(2i+4) + \underbrace{i(2i+4)+4}_{-2+4i+4}$

$= (2+4i) + (2i+4)(z-i) + (z-i)^2 + 0.$

or $f'(z) = 2z+4, \quad f''(z) = 2, \quad f'''(z) = 0, \quad f^{(n)}(z) = 0 \ n>3.$

$\Rightarrow$ $f(z) = f(i) + \frac{f'(i)}{1!}(z-i) + \frac{f''(i)}{2!}(z-i)^2 + 0$

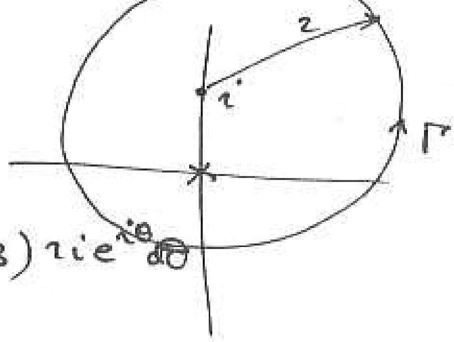
$= (i^2+4i+3) + (2i+4)(z-i) + \frac{2}{2}(z-i)^2$

$= (2+4i) + (2i+4)(z-i) + (z-i)^2 + 0$

(3) (a) $f(z)$ must be holomorphic inside and on Γ.

Γ must be a simple closed curve.

z_0 must be inside Γ.



(b) On Γ $\quad z = i + 2e^{i\theta}$

so $I = \int_\Gamma (2z+3)\,dz = \int_0^{2\pi} \left(2(i+2e^{i\theta})+3\right)2ie^{i\theta}\,d\theta$

$$= \int_0^{2\pi} (2i + 4e^{i\theta} + 3)\,2ie^{i\theta}\,d\theta.$$

$$= 2i(2i+3)\int_0^{2\pi} e^{i\theta}\,d\theta + 8i\int_0^{2\pi} e^{2i\theta}\,d\theta$$

now $\int_0^{2\pi} e^{i\theta}\,d\theta = \int_0^{2\pi} (\cos(\theta) + i\sin\theta)\,d\theta = 0$

& $\int_0^{2\pi} e^{2i\theta}\,d\theta = \frac{1}{2i} e^{2i\theta}\Big|_0^{2\pi} = \frac{1}{2i}\left(e^{4i\pi} - e^{0}\right) = \frac{1}{2i}(1-1) = 0$

Hence $I = 0$.

$J = \int_\Gamma \frac{dz}{z^2}$. deform the contour so it is a circle about $z=0$, Γ_1 the only singularity, a double pole so $z = e^{i\theta}$ on Γ_1

$$= \int_{\Gamma_1} \frac{dz}{z^2}$$

$$= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{2i\theta}} = i\int_0^{2\pi} e^{-i\theta}\,d\theta$$

$$= i\int_0^{2\pi} (\cos\theta - i\sin\theta)\,d\theta = 0 \quad \text{as before.}$$

(c) $f(z) = e^{-z^2}\sin(\pi z)$ is entire & i is inside the circle.

Hence $\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-i} = f(i) \Rightarrow$

$$\oint_\Gamma \frac{f(z)}{z-i} = 2\pi i\, f(i) = 2\pi i\, e^{-i^2}\sin(\pi i)$$

$$= 2\pi i\, e^{1}\left(\frac{e^{i\pi i} - e^{-i\pi i}}{2i}\right)$$

$$= \pi e\left(e^{-\pi} - e^{\pi}\right).$$

(4) (a)

$$f(z) = \sum_{n=1}^{\infty} \frac{(4z-i)^n}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{4^n}{n(n+1)} \left(z - \frac{i}{4}\right)^n$$

Hence $\underline{\text{center}} = \frac{i}{4}$ & $a_n = \frac{4^n}{n(n+1)}$

& $R_f = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\cancel{4^n}\,\cancel{(n+1)}(n+2)}{n\cancel{(n+1)}\,4^{\cancel{n}+1}}$

$$= \lim_{n\to\infty} \left(\frac{n+2}{n}\right)\cdot\frac{1}{4}$$
$$= \lim_{n\to\infty} \left(1+\frac{2}{n}\right)\cdot\frac{1}{4} = \frac{1}{4}$$

Hence $R_f = \frac{1}{4}$

& the circle of convergence is $\left\{z \in \mathbb{C} : \left|z - \frac{i}{4}\right| = \frac{1}{4}\right\}$.

Inside the circle the series converges absolutely. Outside it diverges.

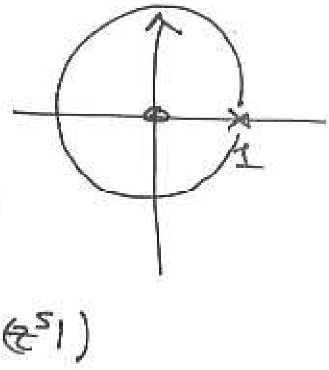
On the circle $z = \frac{i}{4} + \frac{1}{4}e^{i\theta}$

$$f(z) = \sum_{n=1}^{\infty} \frac{(i + e^{i\theta} - i)^n}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n(n+1)} = \sum_{n=1}^{\infty} c_n$$

& $|c_n| = \frac{1}{n(n+1)} < \frac{1}{n^2}$

& $\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty \Rightarrow$ the series $\underline{\text{converge absolutely}}$ at each point on the circle of convergence.

(b) $f(z) = \frac{1+z^2}{1-z}$



$\underline{R_f = 1}$ = min dist to the singularity from the center

$$f(z) = (1+z^2)(1+z+z^2+z^3+z^4+O(z^5))$$
$$= 1 + z + z^2(1+1) + z^3(1+1) + \dots \qquad |z|<1.$$
$$= 1 + z + 2z^2 + 2z^3 + \dots$$