UNIVERSITY OF WAIKATO
Department of Mathematics

MATH311A - Advanced Calculus

TEST 2

Friday 25 May 2007 – 3.10pm

Time allowed: 1 hour
Answer ANY 3 questions
Each question is worth 12 marks out of a total of 36 marks

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1. (a) Let $\rho = e^{2\pi i/3}$. Find the real and imaginary parts of ρ. Show where 1, ρ and ρ^2 sit in the complex plane and verify that $1+\rho+\rho^2=0$.

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(b) Factor $p(z)=z^3+1$ over $\mathbb{C}$ into linear factors [Hint: $p(-1)=0$] and say why this is a case of the Fundamental Theory of Algebra.

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(c) Let $f(z)=\dfrac{1}{(z^3+1)z^2}$. Sketch its singularities in $\mathbb{C}$ and describe their types.

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(d) Expand $f(z)$ in a Laurent Series about $z=0$ $\left[\text{Hint: write } f(z)=\dfrac{1}{z^2\left(1-(-z^3)\right)}\right]$ as far as the term in z^4, and describe an annulus, with centre $z=0$, in which the full Laurent series is valid.

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2. (a) Let $f(z)=u+iv$ be holomorphic on $\Omega\subset\mathbb{C}$ where Ω is open. State the Cauchy-Riemann equations. Use them to show that $f(z)=|z|^2$ is not holomorphic.

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(b) Use the Cauchy-Riemann equations to show that $f(z)=z^2+4z+3$ is holomorphic on $\mathbb{C}$.

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(c) Assuming $f'(z)=u_x+iv_x$ show that for $f(z)$ as defined in (b), $f'(z)=2z+4$.

3

(d) Expand $f(z)$ as defined in (b) in a power series with center $z=i$ to 4 terms [Hint: write $f(z)$ as a polynomial in $(z-i)$.]

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3. (a) Cauchy's Integral Theorem states

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$$f(z_0)=\frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}\,dz$$

and his Theorem states

$$0=\oint_\Gamma f(z)\,dz\,.$$

Give the conditions on $f(z)$, z_0 and Γ under which the theorems are true.

(b) Let Γ be a circle center i and radius 2.

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Use parametric evaluation to show $\oint_\Gamma (2z+3)\,dz=0$ and $\oint_\Gamma \frac{dz}{z^2}=0$.

(c) Use Cauchy's Integral formula to evaluate

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$$\oint_\Gamma \frac{e^{-z^2}\sin(\pi z)}{z-i}\,dz$$

when Γ is the circle $z=i+2e^{i\theta}$.

4. (a) Find the centre, radius, and circle of convergence of the power series

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$$f(z)=\sum_{n=0}^{\infty}\frac{(4z-i)^n}{n(n+1)}$$

and examine the convergence or divergence of the series at each point inside the circle, on the circle, outside the circle.

(b) Expand $f(z)=\frac{(1+z^2)}{1-z}$ in a power series about $z=0$, giving the first 4 terms and the n'th term. Use the singularity of f to find the radius of convergence.

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