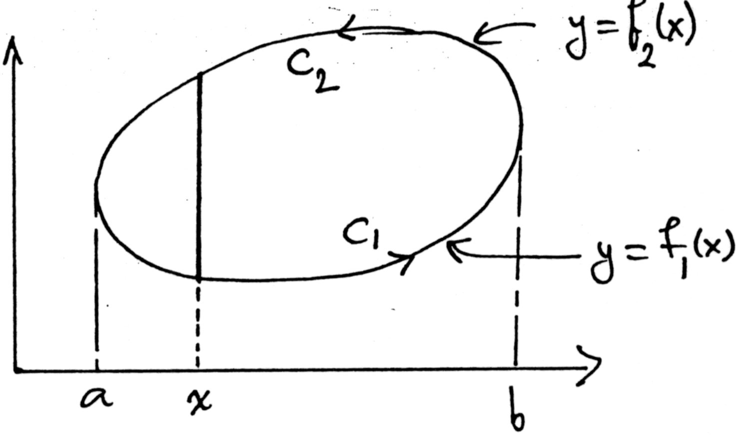
Green's Theorem for a simple region



Fix x in $[a, b]$

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy = M(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x))$$

so

$$\begin{aligned}
\iint_R \frac{\partial M}{\partial y} dydx &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))]\, dx \\
&= -\int_b^a M(x, f_2(x))dx - \int_a^b M(x, f_1(x))dx \\
&= -\int_{C_2} Mdx - \int_{C_1} Mdx = -\int_C Mdx
\end{aligned}$$

therefore

$$-\iint_R \frac{\partial M}{\partial y} dydx = \int_C Mdx$$

Similarly

$$\iint_R \frac{\partial N}{\partial x} dydx = \int_C Ndy$$

therefore

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$