Complex Assignment 4 Solutions

① $f(z) = \dfrac{z+4}{(z+i)^2(z-2)} \Rightarrow \lim_{z\to 2} (z-2)f(z) = \dfrac{2+4}{(2+i)^2} \neq 0$

$= \dfrac{6}{3+4i} = \boxed{\dfrac{6}{25}(3-4i)}$

So @ the simple pole $z = 2$

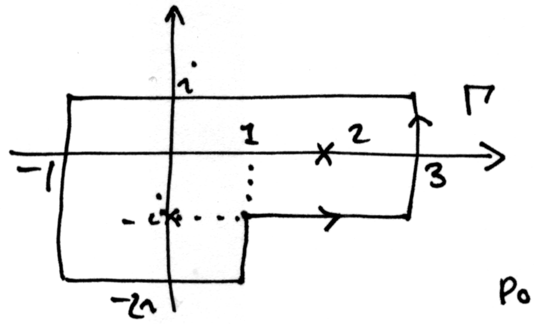
$R_1 = R(f, 2) = \boxed{\dfrac{6(3-4i)}{25}}$

$\lim_{z\to -i} (z+i)^2 f(z) = \dfrac{i+4}{(i-2)^2}$ so $z = -i$ is a double pole

with residue $R_2 = R(f, -i) = \dfrac{1}{(2-1)!} \dfrac{d}{dz}\left((z+i)^2 f(z)\right)\Big|_{z=-i} \Rightarrow$

$R_2 = \dfrac{d}{dz}\left(\dfrac{z+4}{z-2}\right)\Big|_{z=-i} = \dfrac{(z-2)1 - (z+4)1}{(z-2)^2}\Big|_{z=-i} = \dfrac{-6}{(-i-2)^2} = \boxed{-\dfrac{6}{25}(3-4i)}$

②



Poles are at $\{2, -i\}$

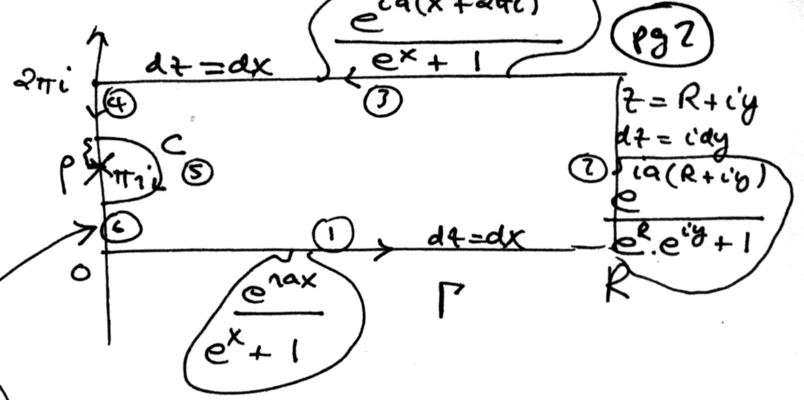
so, by the Residue Theorem, $\int_\Gamma f(z)\,dz = 2\pi i(R_1 + R_2) = 2\pi i(0) = \boxed{0}$ //

③ See Lecture notes p105

④ On the following pages:

#4 Let $\int_0^\infty \frac{\sin(ax)}{e^x+1}\,dx = I$

$$f(z) = \frac{e^{iaz}}{e^z+1}$$



$z = iy$
$dz = i\,dy$

$$\frac{e^{ia(iy)}}{e^{iy}+1}.$$

$\rho \to 0+$

$$\Gamma = \overset{①}{[0,R]} \cup \overset{②}{[R, R+2\pi i]} \cup \overset{③}{[R+2\pi i, 2\pi i]} \cup \overset{④}{[2\pi i, (\pi+\rho)i]} \cup \overset{⑥}{[(\pi-\rho)i, 0]} \cup \overset{⑤}{(\text{semicircle } C)}.$$

Cauchy's Thm $\Rightarrow$

$$0 = \overset{①}{\int_0^R \frac{e^{iax}}{e^x+1}\,dx} + \overset{②}{\int_0^{2\pi} \frac{e^{iaR}\,e^{-ay}}{e^R e^{iy}+1}\,i\,dy} + \overset{③}{\int_R^0 \frac{e^{iax}\,e^{-2\pi a}}{e^x+1}\,dx} + \overset{④}{\int_{2\pi}^{\pi+\rho} \frac{e^{-ay}}{e^{iy}+1}\,i\,dy} + \overset{⑤}{\int_{C_\rho} f(z)\,dz} + \overset{⑥}{\int_{\pi-\rho}^0 \frac{e^{-ay}}{e^{iy}+1}\,i\,dy} \qquad \boxtimes$$

$$①+③ = \int_0^R \frac{\cos(ax)+i\sin ax}{e^x+1}\,dx\left(1-e^{-2\pi a}\right)$$ so we take the imaginary part of $\boxtimes$.

$f(z)$ has a simple pole where $e^z+1=0$, namely $z=\pi i$. There are no other poles inside or on Γ. The residue is $\left.\frac{e^{iaz}}{(e^z+1)'}\right|_{z=\pi i} = \frac{e^{-a\pi}}{e^{\pi i}}$

so the contribution from the pole on the contour as $\rho\to 0+$ $\quad -\pi i(\text{residue}) = \pi i e^{-a\pi}$ (− since we go clockwise).

Hence ⑤ $= \pi e^{-a\pi}$ (imag. pt).

For ② $\quad |f(z)| = \left|\frac{e^{iaR}\,e^{-ay}}{e^R e^{iy}+1}\right| \leq \frac{1}{e^R-1} \to 0$ rapidly

so ② $\to 0$ as $R\to\infty$.

& we set ② $= 0$.

For 4 & 6 : $\displaystyle\int_\alpha^\beta \frac{e^{-ay}}{e^{iy}+1}\, i\,dy = \int_\alpha^\beta \frac{e^{-ay}\quad i}{\cos y + 1 + i\sin y}\,dy$

$$= \int_\alpha^\beta \frac{e^{-ay}\, i\,(\cos y + 1 - i\sin y)}{(\cos y+1)^2 + \sin^2 y}\,dy$$

$$= \int_\alpha^\beta \frac{e^{-ay}\big(i(\cos y+1) + \sin y\big)\,dy}{2 + 2\cos y}$$ and we want only the imaginary part

$$\mathrm{Im}\left(\int_\alpha^\beta \cdots\right) = \int_\alpha^\beta \frac{e^{-ay}}{2}\,dy = \frac{1}{2a}\left(e^{-a\alpha} - e^{-a\beta}\right)$$

Hence $\mathrm{Im}(4 + 6) = \frac{1}{2a}\left(e^{-a2\pi} - a^{-a(\pi+\rho)}\right) + \frac{1}{2a}\left(e^{-a(\pi-\rho)} - e^{0}\right)$

$$\xrightarrow{\rho\to 0+} \frac{1}{2a}\left(e^{-2\pi a} - 1\right)$$

Hence, by $\boxtimes$, $0 = I(1-e^{-2\pi a}) + 0 + \pi e^{-a\pi} + \frac{1}{2a}\left(e^{-2\pi a} - 1\right)$

& letting $\rho\to 0+$, $R\to\infty$

$\Rightarrow$ solving $\displaystyle I = \frac{1}{2a}\,\frac{1-e^{-2\pi a}}{1-e^{-2\pi a}} - \frac{\pi e^{-a\pi}}{1-e^{-2\pi a}}$

$$= \frac{1}{2a} - \frac{2\pi}{2(e^{\pi a} - e^{-\pi a})}$$

$$= \frac{1}{2a} - \frac{\pi}{2\,\mathrm{sh}(\pi a)} \quad ///$$