Advanced Calculus math311-04A 2004

Complex Assignment 3 Solutions

1. Taylor's Theorem $\Rightarrow e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$ where

$f(z) = e^z$. But $f'(z) = f(z) \Rightarrow f^{(n)}(0) = e^0 \ \forall n$ so

$$e^z = \sum_{n=0}^{\infty} \frac{1.z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \boxed{1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i} \sum_{n=0}^{\infty} \left(\frac{(iz)^n}{n!} + - \frac{(-i)^n z^n}{n!} \right)$$

But for n even $(-1)^n = 1 \Rightarrow$ the n'th term vanishes. So let n is odd. &

$n = 2m+1$, $m = 0, 1, 2, \ldots$ so

$i^n = i^{2m+1} = i(i^2)^m = i(-1)^m$ & $(-i)^n = -i(-1)^m$

so
$$\sin(z) = \frac{1}{2i} \sum_{m=0}^{\infty} \frac{(i^n - (-i)^n) z^n}{n!}$$
$$= \frac{1}{2i} \sum_{m=0}^{\infty} \frac{(-1)^m (i - -i) z^{2m+1}}{(2m+1)!}$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!} = \boxed{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$e^z \sin(z) = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right) \times$$
$$\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)$$

picking out the powers of $z \longrightarrow$

$$= z + z^2 + z^3\left(\frac{1}{2} - \frac{1}{6}\right) + z^4\left(\frac{1}{6} - \frac{1}{6}\right) + \cdots$$
$$= \boxed{z + z^2 + \frac{1}{3} z^3 + 0.z^4 + \cdots}$$

Since e^z and $\sin(z)$ are analytic on $\mathbb{C}$, so is $e^z . \sin(z)$.

Hence there are no singularities, hence $R_f = +\infty$.

(2)

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n (z-(-4))^n}{n\,5^n} \Rightarrow \boxed{\text{centre is } -4}$$

$$a_n = \frac{(-1)^n}{n5^n} \quad \text{so} \quad \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)5^{n+1}}{n\,5^n}$$

$$\Rightarrow R_f = \lim_{n\to\infty} \left(\frac{n}{n}+\frac{1}{n}\right)5 = 5. \quad \text{Hence } \boxed{R_f = 5}$$

& circle of convergence is $\{z : |z-(-4)| = 5\}$.

On the circle $z = -4 + 5e^{i\theta}$ so

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n e^{ni\theta} 5^n}{n\,5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n e^{ni\theta}}{n}$$

At $\theta = 0$ $f(-4+5e^{i.0}) = f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is a real alternating series so it converges (conditionally)

At $\theta = \pi$ $f(-4+5e^{i\pi}) = f(-9) = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n}$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

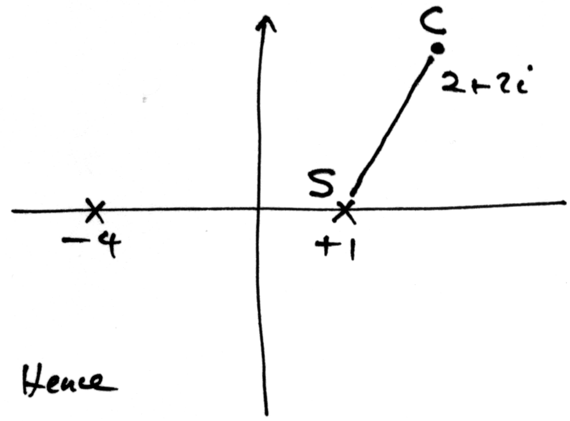
If $\theta \neq \pi$ $f(-4+5e^{i\theta})$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n e^{in\theta}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n}$$

and, although neither is alternating, both of these series, i.e. the real and imaginary parts, converge (proof not given).

Note that $\left|\frac{(-1)^n e^{in\theta}}{n}\right| = \frac{1}{n}$ so $f(z)$ does not converge absolutely at any point on the circle.

(3) $$f(z) = \frac{z}{(z-1)(z+4)}$$



Then f has singularities at $z=1$ and $z=-4$. The closest to the centre C is at $S=1$. Hence the radius of convergence $R_f = |2+2i-1| = |1+2i|$

$$= \sqrt{1^2+2^2} = \sqrt{5}.$$

(4) $$f(z) = \frac{z^3\sqrt{z+4}}{(z+i)^2(z-2)}$$

(a) $z=0$: zero of order 3

(b) $z=-4$: f is not defined, branch point singularity.

(c) $z=-i$: pole of order 2

(d) $z=2$: pole of order 1 or a simple pole.