Math 311 – 03A Advanced Calculus

Complex Assignment 2 Solutions

(1)
$$f(z) = 3z^2 - 8z + 1$$
$$f(x+iy) = 3(x+iy)^2 - 8(x+iy) + 1$$
$$= 3(x^2-y^2+2ixy) - 8x - 8iy + 1$$
$$= 3(x^2-y^2) - 8x + 1 + i(6xy - 8y)$$
$$\Rightarrow \quad u = 3x^2 - 3y^2 - 8x + 1 \quad \& \quad v = 6xy - 8y$$

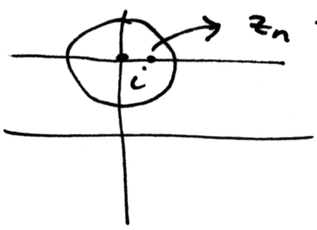
so
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 6x - 8 & \frac{\partial v}{\partial x} &= 6y \\ \frac{\partial u}{\partial y} &= -6y & \frac{\partial v}{\partial y} &= 6x - 8 \end{aligned}\right\} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at every point $(x,y) \in \mathbb{R}^2$. Hence f is holomorphic on $\mathbb{C}$.

now
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= 6x - 8 + i6y = 6(x+iy) - 8 = 6z - 8.$$

(2) If $\lim_{z \to i} \frac{2}{z-i} = L \in \mathbb{C}$ let $\varepsilon = 1$

$\exists\, \delta_1 > 0$ so $0 < |z - i| < \delta_1 \Rightarrow \left|\frac{2}{z-i} - L\right| < 1$ (α)



$z_n = i + \frac{1}{n}$

Choose $z_n = i + \frac{1}{n}$ so $|z_n - i| < \delta_1 \ \forall n \geq N_1$.

Then (α) $\Rightarrow \left|\frac{2}{1/n} - L\right| < 1$

By the extension of the Δ Law $\left|\frac{2}{1/n}\right| - |L| \leq \left|\frac{2}{1/n} - L\right| < 1$

$$\Rightarrow 0 < 2n < 1 \quad \forall n \geq N_1$$

which is impossible.

Hence the limit does not exist.

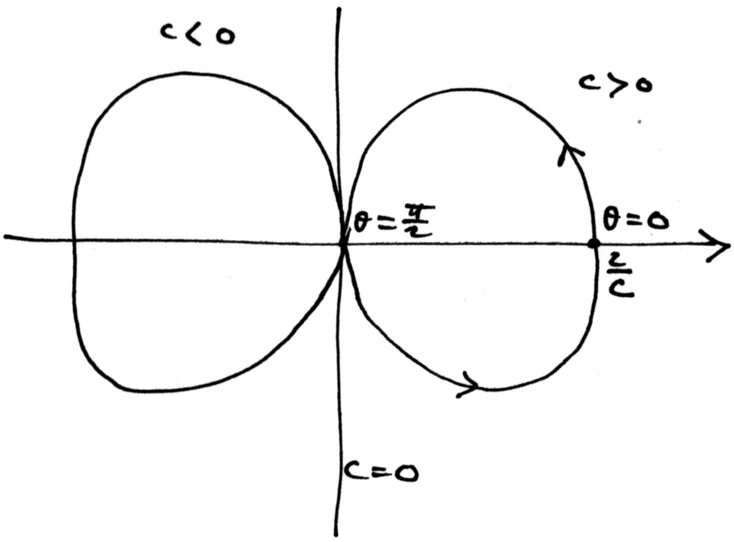
$$f(z) = \frac{2}{z-i} = \frac{2}{x+(y-1)i} = \frac{2(x - i(y-1))}{x^2+(y-1)^2} \text{ so } u = \frac{2x}{x^2+(y-1)^2} \ (\beta)$$
$$= u + iv \qquad v = \frac{-2(y-1)}{x^2+(y-1)^2} \ (\gamma)$$

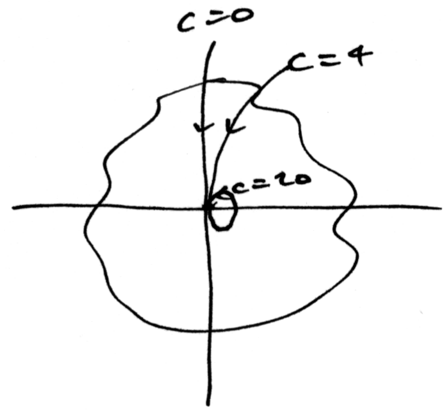
The easiest way to examine the contours is to shift i to $(0,0)$ & then use polar coordinates $x = r\cos\theta$ $y = r\sin\theta$

e.g. for $u \longrightarrow$ $u = \dfrac{2x}{x^2+y^2} = \dfrac{2r\cos\theta}{r^2} = \dfrac{2\cos\theta}{r}$

so $u = \text{constant} = c$ on a contour $\Rightarrow$ $c = u = \dfrac{2\cos\theta}{r}$

so $r = \frac{2}{c}\cos(\theta)$. This curve is a circle of diameter $\frac{2}{c}$





If $c=0$, $c = u = \dfrac{2x}{x^2+y^2}$ $\Rightarrow$ $x=0$ so the contour is the y-axis. So as you 'come into 0' along a contour u has the value c. But near 0 there are contours with every possible $\mathbb{R}$ value. This goes to show how <u>nasty</u> the function $f(z)$ is in every neighbourhood of i. //

(3) Let $\log z_1 = \omega_1$, $\log z_2 = \omega_2$ so $e^{\omega_1} = z_1$, $e^{\omega_2} = z_2$

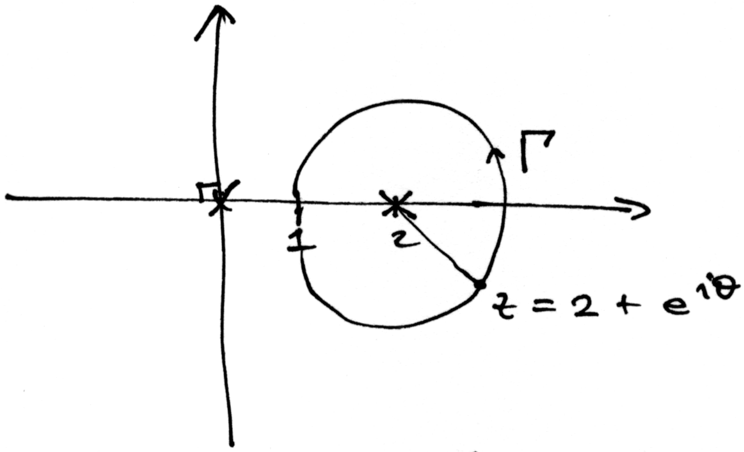
& $z_1 z_2 = e^{\omega_1} \cdot e^{\omega_2} = e^{\omega_1+\omega_2}$ so $\log(z_1z_2) = \omega_1 + \omega_2 = \log z_1 + \log z_2$

<u>OR</u> $\log(z) = \log|z| + i\arg z \Rightarrow$

$\log(z_1) = \log|z_1| + i\arg z_1$, $\log(z_2) = \log|z_2| + i\arg z_2$

$$\begin{aligned}\Rightarrow \log(z_1) + \log(z_2) &= \log|z_1| + \log|z_2| + i(\arg z_1 + \arg z_2)\\ &= \log(|z_1|\cdot|z_2|) + i\arg(z_1z_2)\\ &= \log|z_1z_2| + i\arg(z_1z_2)\\ &= \log(z_1z_2)\ //\end{aligned}$$

(4)



$$\frac{1}{z^2-2z} = \frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

so $\quad 1 = A(z-2) + Bz \quad \Rightarrow \quad A = -\frac{1}{2} \ (z=0)$, $\quad B = \frac{1}{2} \ (z=2)$

Hence $I = \int_\Gamma \frac{dz}{z^2-2z} = -\frac{1}{2}\int_\Gamma \frac{dz}{z} + \frac{1}{2}\int_\Gamma \frac{dz}{z-2}$

now $\frac{1}{z}$ is analytic <u>inside</u> and <u>on</u> Γ, hence, by Cauchy's Theorem, $\int_\Gamma \frac{dz}{z} = 0$.

By Cauchy's integral formula with $f(z) = 1$, since $z=2$ is <u>inside</u> Γ,

$1 = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-2}$. Hence $I = 0 + \frac{1}{2} \cdot 2\pi i = \pi i$ //

Complex <u>parametric</u> evaluation: $z = 2 + e^{i\theta}$ on Γ. Hence

$$I = \int_0^{2\pi} \frac{i e^{i\theta}\, d\theta}{(2+e^{i\theta})e^{i\theta}} = i\int_0^{2\pi} \frac{d\theta}{2+e^{i\theta}}$$

$$= i\int_0^{2\pi} \frac{(2+e^{-i\theta})\, d\theta}{(2+e^{i\theta})(2+e^{-i\theta})}$$

$$\int_0^{2\pi} \frac{\sin(\theta)\, d\theta}{(2+\cos\theta)^2+\sin^2\theta} + i\int_0^{2\pi} \frac{2+\cos\theta}{5+2\cos\theta} \longleftarrow = i\int_0^{2\pi} \frac{2+\cos\theta - i\sin\theta}{|2+\cos\theta + i\sin\theta|^2}$$

$$= \int_0^{2\pi} \frac{\sin\theta\, d\theta}{5+2\cos\theta} = 0$$

an <u>odd</u> funct of θ.

$$\longrightarrow = i\int_0^{2\pi} \frac{2+\cos(\theta)}{5+2\cos(\theta)}\, d\theta$$

a v. difficult integral, best approached using $z = e^{i\theta}$!!