(1) $f(z) = \dfrac{z+2}{z-3} = \dfrac{z-3+5}{z-3} = 1 + \dfrac{5}{z-3} = 1 + \dfrac{5}{(x-3)+iy} \Leftarrow z = x+iy$

$= 1 + \dfrac{5(x-3-iy)}{(x-3)^2+y^2}$

$= u + iV \Rightarrow \quad u = 1 + \dfrac{5(x-3)}{(x-3)^2+y^2}$

$v = \dfrac{-5y}{(x-3)^2+y^2}$

$\Rightarrow \quad u_x = 0 + \dfrac{5}{(x-3)^2+y^2} - \dfrac{5(x-3)2(x-3)}{((x-3)^2+y^2)^2}$

$V_x = \dfrac{10y(x-3)}{((x-3)^2+y^2)^2} \Rightarrow$

$f'(z) = u_x + iV_x$

$= \dfrac{5}{(x-3)^2+y^2} - \dfrac{10(x-3)^2}{((x-3)^2+y^2)^2} + \dfrac{i\,10y(x-3)}{((x-3)^2+y^2)^2}$ //

(No need to simplify to an expression in z unless asked to do so.)

(2)
$u = 6x^2 + 5xy - 6y^2$

$u_x = 12x + 5y$

$u_{xx} = 12$

$u_y = 5x - 12y$

$u_{yy} = -12$

$\Rightarrow \quad u_{xx} + u_{yy} = 12 - 12 = 0$

$\Rightarrow \quad u$ is harmonic (on $\mathbb{R}^2$).

$V_y = u_x = 12x + 5y \Rightarrow \quad V = 12xy + \dfrac{5y^2}{2} + f(x)$

$-V_x = u_y = 5x - 12y \Rightarrow \quad V_x = -5x + 12y = 12y + f'(x)$

$\Rightarrow \quad -5x = f'(x)$

$\Rightarrow \quad f(x) = -\dfrac{5x^2}{2} + c$

Let $c = 0 \Rightarrow f(z) = u + iV$

$= 6x^2 + 5xy - 6y^2 + i\left(\dfrac{5y^2}{2} - \dfrac{5x^2}{2} + 12xy\right)$

$= 6(x^2 - y^2 + 2ixy) + \dfrac{5i}{2}(y^2 - x^2 - 2ixy)$

$= 6z^2 - \dfrac{5i}{2}z^2 = z^2(6 - 5i/2)$ //

(3)

$$\left.\begin{array}{l} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{array}\right\} \Rightarrow \quad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\begin{aligned} \log(z_1 z_2) &= \log(r_1 r_2) + i(\theta_1+\theta_2) \\ &= \log r_1 + \log r_2 + i\theta_1 + i\theta_2 \\ &= (\log r_1 + i\theta_1) + (\log r_2 + i\theta_2) \\ &= \log z_1 + \log z_2 \quad // \end{aligned}$$

(4)

$$g(z) = u + iv$$
$$f(u+iv) = A + iB$$

$$\begin{aligned} \text{lhs} &= (f\circ g)'(z) = A_x + iB_x \\ &= (A_u u_x + A_v v_x) + i(B_u u_x + B_v v_x) \end{aligned}$$

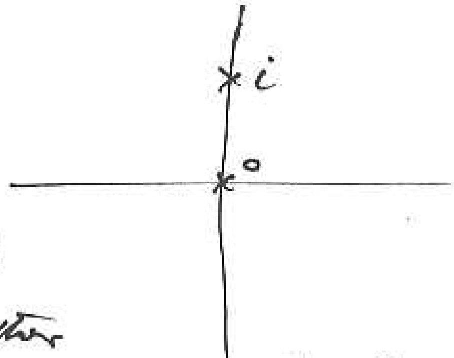
$$\overline{\overline{A_u = B_v}}$$

$$\begin{aligned} \text{rhs} = f'(g(z))\cdot g'(z) &= (A_u + iB_u)(u_x + iv_x) \\ &= (A_u u_x - B_u v_x) + i(A_u v_x + B_u u_x) \end{aligned}$$

$$\underline{A_v = -B_u}$$

so lhs = rhs by C-R. //

(5)



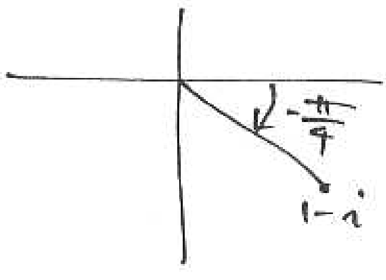
$$w^2 = z(z-i)$$

$$\Delta \arg w = \tfrac{1}{2}\Delta \arg z + \tfrac{1}{2}\Delta \arg(z-i)$$

so any path going about 0 or i & not the other changes the sign of w. Hence a domain is $\mathbb{C} \setminus [0,i]$

$$= \{z \in \mathbb{C} : \operatorname{Im}(z) \notin [0,1]\}$$

$$w^2 = z(z-i) \Rightarrow w_1^2 = 1(1-i)$$
$$= 1 - i = re^{i\theta} \qquad r = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\theta = -\pi/4$$



$$\text{so } w_1^2 = \sqrt{2}\, e^{-i\pi/4}$$
$$w_1 = 2^{1/4} e^{-i\pi/8} \quad //$$