math 311 - 03A Advanced Calculus

Complex Assignment 1 Solutions

(1) If $a_n = n i^n$, by the Cauchy condition for convergence with $\varepsilon = 1$, if (a_n) converges then $\exists\, N_1$ so $\forall\, n, m \geq N_1$

$|a_n - a_m| < 1$. But this is true $\Leftrightarrow$

$|n i^n - m i^m| < 1$

But $\big|\,|n i^n| - |m i^m|\,\big| \leq |n i^n - m i^m|$ by Δ law

so $|n - m| < 1$ if $n = N_1 + 1$ and $m = N_1$

we get $|(N_1+1) - N_1| < 1 \Leftrightarrow 1 < 1$ which is false.

Hence (a_n) does not converge.

(2) $$L = \lim_{n\to\infty} \frac{(2n+4i-3)(n-1)}{in^2 - in + 1 - 3i} = \lim_{n\to\infty} \frac{\frac{2n+4i-3}{n}\cdot\frac{n-1}{n}}{\frac{in^2 - in + 1 - 3i}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\left(2 + \frac{4i-3}{n}\right)\left(1 - \frac{1}{n}\right)}{\left(i - \frac{i}{n} + \frac{1-3i}{n^2}\right)} = \frac{\left(\lim_{n\to\infty} 2 + \frac{4i-3}{n}\right)\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right)}{\lim_{n\to\infty}\left(i - \frac{i}{n} + \frac{1-3i}{n^2}\right)}$$

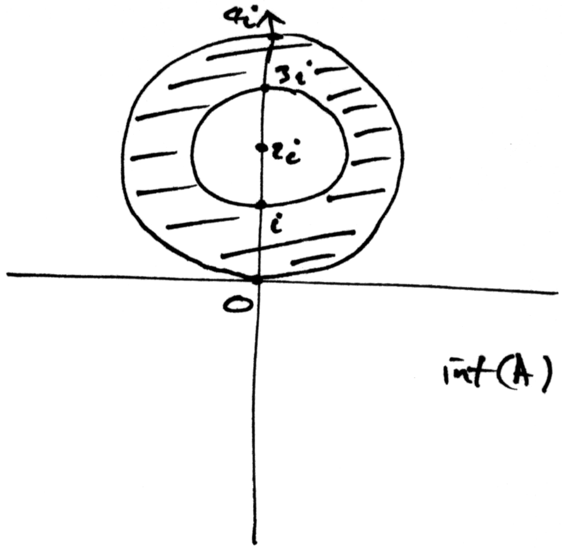
using the lt. of a quotient is the quotient of the limits.

now each of these individual limits can be computed using the lt. of a sum is the sum of the limits and $\frac{1}{n} \to 0$ e.g.

$$\lim_{n\to\infty}\left(2 + \frac{4i-3}{n}\right) = \lim_{n\to\infty} 2 + (4i-3)\lim_{n\to\infty}\frac{1}{n} = 2 + (4i-3)\cdot 0$$
$$= 2$$

Hence $$L = \frac{(2+0)(1-0)}{i - i\cdot 0 + (1-3i)\cdot 0} = \frac{2}{i} = -2i$$

③



$$\operatorname{int}(A) = A^{\circ} = \{z : 1 < |z-2i| < 2\} = A$$

$$\bar{A} = \{z : 1 \le |z-2i| \le 2\}$$

$$\partial A = \{z \in \mathbb{C} : |z-2i| = 1\} \cup \{z \in \mathbb{C} : |z-2i| = 2\}$$

A is open and connected $\Rightarrow$ A is a region or domain.
∂A is not connected, since for example $0 \in \partial A$, $i \in \partial A$ and these points cannot be joined by a continuous path lying entirely in A.

④ Let $A = \{z : \operatorname{Re}(z) = 0\}$ i.e. the imaginary axis.
Then A is <u>unbounded</u> since $\{ni : n \in \mathbb{N}\} \subset A$ & $|ni| = n \to \infty$.

But $|f(z)| = |e^z| = |e^x||e^{iy}| = e^x$

so $|f(iy)| = e^0 = 1$ and $|f|$ is <u>bounded</u> on A.

(Indeed any unbounded subset of $A_\sigma = \{z : \operatorname{Re}(z) \le \sigma\}$ for any $\sigma \in \mathbb{R}$ has the given property.)

⑤ Given $\varepsilon > 0$. Want $\delta_\varepsilon > 0$ so $0 < |z-1-i| < \delta_\varepsilon \Rightarrow |f(z) - (2i+1)| < \varepsilon$.

The latter is true $\Leftrightarrow |z^2+1-2i-1| < \varepsilon$
$\Leftrightarrow |z^2-2i| < \varepsilon$
$\Leftrightarrow |z-1-i| \cdot |z+(1+i)| < \varepsilon$ ⊛

If $\delta_1 = 1$ then $|z-(1+i)| < 1 \Rightarrow |z+1+i| = |z-(1+i)+2(1+i)|$
$\le |z-(1+i)| + 2|1+i|$
$< 1 + 2\sqrt{2}$

Hence chose $\delta_\varepsilon = \min\left\{\dfrac{\varepsilon}{1+2\sqrt{2}}, 1\right\}$

<u>Note</u> ⊛ $z^2 - 2i = (z-\alpha)(z-\beta)$ $\quad \alpha, \beta = \dfrac{0 \pm \sqrt{0+8i}}{2}$ & $\sqrt{i} = \dfrac{1+i}{\sqrt{2}}$
$= \pm(1+i)$ //