

**The University of Waikato Department of  
Mathematics**

**Elements of Analysis Workshop/Tutorial for 30th September: math252-09B**

For the Workshop/Tutorial on 30th September, attempt questions 1-6.

Name: \_\_\_\_\_ Id: \_\_\_\_\_

**Uniform convergence of sequences and series of functions:**

1. For each  $n \in \mathbb{N}$  and  $x \in \mathbb{R}$  let

$$f_n(x) = \frac{x}{n} + x.$$

Show that the sequence of functions  $(f_n : n \in \mathbb{N})$  converges pointwise to the function  $h(x) = x$  on  $\mathbb{R}$ .

2. Now show the convergence of the sequence in 1. is uniform on  $[0, 1]$  and then  $[0, 2]$

3. Keeping to this same sequence show the convergence is uniform on  $[-c, c]$  for every real  $c > 0$ .

4. Now verify each  $f_n$  is continuous and deduce  $h$  is continuous on  $\mathbb{R}$ , even though the convergence is not uniform on  $\mathbb{R}$ . To show it is not uniform try  $\epsilon = 1$  and test  $|f_n(x) - h(x)| < 1$  at  $x = N_\epsilon + 1$ .

5. Show that  $f'_n \rightarrow h_1$ , where  $h_1(x) = 1$  for all  $x$ , and that the convergence is uniform on  $[0, 2]$ . Hence verify that  $\lim_{n \rightarrow \infty} f'_n = (\lim_{n \rightarrow \infty} f_n)'$ . Then show this explicitly.

6. Let  $g_n(x) = nx$  for  $0 \leq x \leq 1/n$  and  $g_n(x) = n(1-x)/(n-1)$  for  $n \in \mathbb{N}$  and  $x \in [0, 1]$ . Verify pointwise convergence to  $h_0$ , where  $h_0(x) = 0$  for all  $x$ , and then show, by explicit integration of  $g_n$  that this convergence is not uniform. Draw some pictures then to show what is going on.

30th September 2009