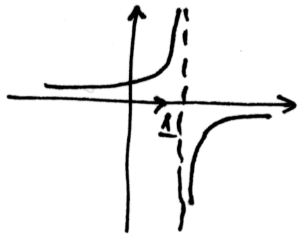
math 252-04A Introd. to Real Analysis 2004

Test 2 Solutions



$y = \dfrac{1}{1-x}$

① (a) $L = \lim\limits_{x \to 1-} \dfrac{1}{1-x} = \boxed{\infty}$

Let $h > 0$ $x = 1-h$

then $x \to 1- \Leftrightarrow h \to 0+$

& $L = \lim\limits_{h \to 0+} \dfrac{1}{h} = \infty.$

(b) $\lim\limits_{x \to -2} \dfrac{x^2-4}{x+2} = \lim\limits_{x \to -2} \dfrac{(x+2)(x-2)}{x+2} = \lim\limits_{x \to -2} (x-2) = -2-2 = \boxed{-4}$

(c) $\lim\limits_{x \to \infty} \dfrac{x^2-1}{x^2+x+1} = \lim\limits_{x \to \infty} \dfrac{1-\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{x^2}} = \dfrac{1-0}{1+0+0} = \boxed{1}$

(d) $\lim\limits_{x \to 0+} \dfrac{\sin(3x)}{\cancel{\sin} x} = \lim\limits_{x \to 0+} \dfrac{3 \sin 3x}{3x} = 3 \lim\limits_{\theta \to 0+} \dfrac{\sin(\theta)}{\theta} = 3.1 = \boxed{3}$

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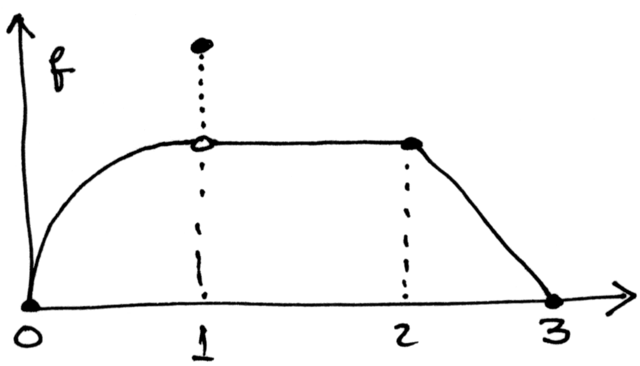
②



(a) f is continuous and differentiable on $(0,1) \cup (1,2) \cup (2,3)$.

(b) At $x = 0$ f is continuous on the right but $f_+'(0) = +\infty$ so it is not diff on the right.

(c) At $x = 1$ $\lim\limits_{x \to 1} f(x) = 1 \neq f(1) = 2$ so f is not cont $\therefore$ not diff.

(d) At $x = 2$ $f_+'(2) = -1 \neq 0 = f_-'(2)$ so f is not diff.

However it is continuous.

(e) At $x = 3$ $\lim\limits_{x \to 3-} f(x) = 0 = f(3)$ & $f_-'(3) = -1$ so it is continuous and diff on the left at $x = 3$. //

(3) (a) $\lim\limits_{x\to x_0-} f(x) = L \iff \forall \varepsilon>0\ \exists \delta_\varepsilon>0$ such that

$$x_0-\delta_\varepsilon < x < x_0 \Rightarrow |f(x)-L|<\varepsilon.$$

(b) $\lim\limits_{x\to x_0+} f(x) = -\infty \iff \forall M>0\ \exists \delta_M$ such that

$$x_0 < x < x_0+\delta_M \Rightarrow f(x) < -M.$$

(c) f is differentiable at $x=x_0 \iff \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}\ \exists.$

(d) $f_+'(x_0) = L \iff \forall\varepsilon>0\ \exists\delta_\varepsilon>0$ such that $x_0<x<x_0+\delta_\varepsilon \Rightarrow$

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - L\right| < \varepsilon.$$

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(4) Given $\varepsilon>0$ want $|f(x)-4|<\varepsilon$. This is true

$\iff |(2x^2-4)-4| < \varepsilon$

$\iff 2|x^2-4|<\varepsilon \iff 2|x+2|\cdot|x-2|<\varepsilon$ ⊠

Let $\delta_1 = 1$ then $2<x<2+\delta_1 \Rightarrow 2<x<3$ &

$|x+2| = x+2 < 5$. So if $2.5.|x-2|<\varepsilon$ we would be done. To this end let $\delta_\varepsilon = \min\{1, \frac{\varepsilon}{10}\}$.

Then $2<x<2+\delta_\varepsilon \Rightarrow 2<x<3$ & $2<x<2+\frac{\varepsilon}{10}$ so

$|x+2|<5$ and $|x-2|<\frac{\varepsilon}{10}$

So $2|x+2|.|x-2| < 2.5.\frac{\varepsilon}{10} = \varepsilon$, so by ⊠

$|f(x)-4|<\varepsilon$. Hence $\lim\limits_{x\to 2+} f(x) = 4$.

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(5) Cont at $x=2$: $\lim\limits_{x\to2-} f(x) = \lim\limits_{x\to2-} ax = 2a = \lim\limits_{x\to2+} f(x)$

$= 4a-2b+3$

Hence $\boxed{2b = 2a+3}$

Diff at $x=2$: $f_-'(2) = f_+'(2) \Rightarrow a = 2a.2 - b \Rightarrow \boxed{b=3a}$

solving □+□ for a and b gives $a = 3/4,\ b = \frac{9}{4}$ //.

6\. (a) Since f is continuous at $x = c$,

$\lim_{x \to c} f(x) = f(c)$. Let $\varepsilon = \frac{f(c)}{2} \Rightarrow \exists \delta_\varepsilon > 0$

such that $c - \delta_\varepsilon < x < c + \delta_\varepsilon \Rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$. But

then $f(c) - f(x) \le |f(c) - f(x)| = |f(x) - f(c)| < \frac{f(c)}{2}$

$$\Rightarrow \quad \frac{f(c)}{2} = f(c) - \frac{f(c)}{2} < f(x) \quad \text{so}$$

$0 < f(x)$ on $(c - \delta_\varepsilon, c + \delta_\varepsilon)$. //

(b) Bookwork — see your notes.

7\. If f has a derivative of order $N+1$ on an open interval I including x_0 and $x \in I$, then there exist a $\xi \in I$, with ξ between x and x_0, such that

$$f(x) = f(x_0) + \sum_{j=1}^{N} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1}$$

If $x_0 = 0$ & $N = 3$ and $f(x) = \frac{1}{1+x}$:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \frac{24}{4!\,(1+\xi)^5} x^4$$

for some ξ, depending on x, between x and 0.

8\. Let $f(x) = \frac{x+1}{x+2}$ & $M > 0$ be given. Let

$x = -2 + h$ with $h > 0$. Then $x \to -2+ \iff h \to 0+$

& $\lim_{x \to -2+} f(x) = -\infty \iff \lim_{h \to 0+} f(-2+h) = -\infty$. But this is

$$\frac{-2+h+1}{-2+h+2} < -M \iff \frac{-1+h}{h} < -M$$

$$\iff 1 + M < \frac{1}{h}$$

$$\iff h < \frac{1}{M+1}$$

So let $\boxed{\delta_M = \frac{1}{1+M}}$ //