UNIVERSITY OF WAIKATO
Department of Mathematics

MATH252A – Introduction to Real Analysis - 2004

TEST 1

Thursday 29 April 2004 - (50 mins) – Answer any **FOUR** questions

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1. Note whether each of the following sequences converges to a real number or diverges to infinity or diverges. For any that converge, estimate the value of the limit. (There is no need to prove your answer is correct).

   (a) $a_n = \frac{1}{n} + n^2$, (b) $b_n = \frac{n^2+1}{n^3+2}$,

   (c) $c_n = 7 + \frac{(-1)^n}{n+1}$, (d) $d_{n+1} = d_n + 2, d_1 = 0$.

2. (a) Let (a_n) be a sequence of real numbers and let Define, using ε and N_ε, what we mean by the statement

$$\lim_{n\to\infty} a_n = L.$$

   (b) Given $\varepsilon > 0$, find N_ε to prove that if $a_n = \frac{n+7}{4n+5}$, then $\lim_{n\to\infty} a_n = \frac{1}{4}$.

3. (a) Define least upper bound (lub). Let (a_n) be an increasing sequence which is bounded above. Prove

   (b) State a similar theorem which applies to decreasing sequences. Use this to show that if $0 < c < 1$ then $\lim_{n\to\infty} c^n = 0$.

4. (a) Assume that $\lim_{n\to\infty} b_n = M$ and $M \neq 0$. Prove that $\lim_{n\to\infty}\left(\frac{1}{b_n}\right) = \frac{1}{M}$.

   (b) Use this, and other clearly stated limits theorems, to evaluate

$$\lim_{n\to\infty}\left(\frac{1}{6+\frac{1}{n}+\frac{\sin(n)}{n^2}}\right).$$

5. Note, giving a reason in each case, whether each of the following series converges or diverges. (In the case of convergence, do **not** attempt to sum the series).

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{5^n}+\frac{3}{6^n}\right)$, (b) $\sum_{n=1}^{\infty}\frac{1}{n^3+2}$,

(c) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$, (d) $\sum_{n=1}^{\infty}\frac{n}{n^2+2}$.

6. (a) Define the expression $\lim_{x\to x_0^+} f(x) = L$.

(b) Sketch the graph of $y = \frac{x|x-1|}{(x-1)}$ and evaluate $\lim_{x\to 1^+} f(x)$ where $f(x) = \frac{x|x-1|}{(x-1)}$.

Prove your answer is correct using ε and δ_ε.

7. Let $f(x) = 2x^2 + 5$. Given $\varepsilon > 0$ find a $\delta_\varepsilon > 0$ so that if $|x-1| < \delta_\varepsilon$ then $|f(x) - f(1)| < \varepsilon$.

Hence prove f is continuous at $x = 1$.

8. (a) Define the expression $\lim_{x\to x_0^+} f(x) = \infty$

and say in words what it means.

(b) Sketch the graph of $y = \frac{x}{x-2}$. Use ε and δ_ε to prove $\lim_{x\to 2^+}\frac{x}{x-2} = +\infty$.