5) $s_n = a_1 + a_2 + \cdots + a_n \quad \forall n \in \mathbb{N}$.

Defn $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ when this limit exists.

(a) $r = \frac{1}{5^n}$ so $1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1} = s_n$

Hence $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{r^n - 1}{r-1} = \frac{0-1}{r-1} = \frac{1}{1-r} = \frac{1}{1-\frac{1}{5}}$

$= \frac{5}{4}$ //

(b) $\displaystyle\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$

so $s_n = \left(\frac{1}{3} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cdots + \left(\cancel{\frac{1}{n+2}} - \frac{1}{n+3}\right)$

$= \frac{1}{3} - \frac{1}{n+3}$. Hence $\displaystyle\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{n+3}\right)$

$= \frac{1}{3} - 0 = \frac{1}{3}$ //

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6) If $a_n \geq 0$ and $b_n \geq 0$ and $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$ then

If $\sum_{n=1}^{\infty} b_n$ C so does $\sum_{n=1}^{\infty} a_n$. If $\sum_{n=1}^{\infty} a_n$ D so does $\sum_{n=1}^{\infty} b_n$.

(a) $\displaystyle\frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \geq \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ D $\therefore \displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ D.

(b) Here $a_n = \frac{2^n}{n!}$ so $\displaystyle\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$= \frac{2}{n+1} \leq 1/2 < 1$

$\iff 4 \leq n+1 \iff 3 \leq n$.

Hence, by the ratio test, $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

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7) $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n} \quad n \geq 1$. See the

Lecture notes on <u>Monotone Convergence</u>.

8 (a) $\lim_{x \to x_0} f(x) = L$ if

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that if $0 < |x - x_0| < \delta_\varepsilon$ then
$|f(x) - L| < \varepsilon$

(b) Given $\varepsilon > 0$, we want $\delta_\varepsilon > 0$ so

$$|(x^2+1) - 5| < \varepsilon \qquad (*)$$

$$\Leftrightarrow |x^2 - 4| < \varepsilon$$

$$\Leftrightarrow |(x-2)(x+2)| < \varepsilon$$

$$\Leftrightarrow |x-2| \cdot |x+2| < \varepsilon \qquad \text{for } x \text{ near } 2.$$

Let $\delta_1 = 1$ so $|x-2| < 1$. Then

$$|x+2| = |x-2+4| \leq |x-2| + 4 < 1 + 4 = 5$$

So if we have $|x-2| \cdot 5 < \varepsilon$ we could achieve the goal (*). Let $\boxed{\delta_\varepsilon = \min\{1, \frac{\varepsilon}{5}\}}$.

then if $0 < |x-2| < \delta_\varepsilon$ we have $|x-2| < 1$ and
$|x-2| < \frac{\varepsilon}{5}$
$\Rightarrow |x-2| \cdot 5 < \varepsilon$

$$\Rightarrow |x-2| \cdot |x+2| < |x-2| \cdot 5 < \varepsilon$$

so $|(x^2+1) - 5| < \varepsilon$. Hence $\lim_{x \to 2} (x^2+1) = 5$. //