UNIVERSITY OF WAIKATO
Department of Mathematics

MATH252A – Introduction to Real Analysis

TEST 1

Thursday 10 April 2003 - (50 mins) – Answer any FIVE questions

---

1. Note whether each of the following sequences converges or diverges. For any that converge, estimate the value of the limit. (There is no need to prove your answer is correct).

(a) $\dfrac{n^2}{1+n^2}$, (b) $2+(-2)^n$,

(c) $\dfrac{4n^2+2}{2n+1}$, (d) $\dfrac{\sin(n)}{n^3}$.

2. (a) Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$. Define, using ε and N_ε, what we mean by the statement

$$\lim_{n\to\infty} a_n = L.$$

(b) Given $\varepsilon > 0$, find N_ε to prove that if $a_n = \dfrac{4n-1}{n+2}$, then $\lim\limits_{n\to\infty} a_n = 4$.

3. (a) If $\lim\limits_{n\to\infty} a_n = L$ prove, using ε and N_ε, that $\lim\limits_{n\to\infty} 2a_n = 2L$.

(b) Use clearly stated limits theorems to evaluate

$$\lim_{n\to\infty}\left(\frac{5n^3+n^2-7}{2n^3+4n+8}\right)\cdot\left(\frac{2n+1}{n+1}\right).$$

4. Note, giving a reason in each case, whether each of the following series converges or diverges.

(In the case of convergence, do **not** attempt to sum the series).

(a) $\sum\limits_{n=1}^{\infty}\left(\frac{7}{8}\right)^n$, (b) $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+1}$,

(c) $\sum\limits_{n=1}^{\infty}\frac{n^2}{n^3+1}$, (d) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$.

5. Let (a_n) be a sequence and (s_n) the sequence of partial sums. Use this latter sequence to define $\sum_{n=1}^{\infty} a_n$. Sum the following convergent series by first computing the corresponding (s_n):

(a) $\sum_{n=0}^{\infty} \frac{1}{5^n}$,

(b) $\frac{1}{3.4}+\frac{1}{4.5}+\cdots\frac{1}{(n+2)(n+3)}+\cdots$.

6. (a) Give the statement of the "comparison test". Use it to determine whether the series

$$\sum_{n=2}^{\infty} \frac{n+1}{n^2}$$

converges or diverges.

(b) Use the "ratio test" to determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges or diverges.

7. Consider the sequence $(a_n : n \in \mathbb{N})$ defined by $a_1 = 1, a_{n+1} = \sqrt{1+a_n}, n \geq 1$.

(a) prove that the sequence (a_n) is bounded above by 2,

(b) verify that the sequence is monotone increasing,

(c) deduce that the sequence converges and calculate its limit.

8. (a) Define the expression $\lim_{x \to x_0} f(x) = L$.

(b) Use ε and δ_ε to prove that $\lim_{x \to 2} x^2 + 1 = 5$.